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THE THEORY AND PRACTICE OF

ABSOLUTE MEASUREMENTS

IN

ELECTRICITY AND MAGNETISM

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THE THEORY AND PRACTICE OF

ABSOLUTE MEASUREMENTS

IN

ELECTRICITY AND MAGNETISM

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ERRATUM.

Page 778, line 2 from foot, for "parallel" *read* "perpendicular."

CHAPTER VII

GALVANOMETRY AND MEASUREMENT OF CURRENTS

SECTION I

GALVANOMETRY

SINCE currents flowing in a given circuit are taken Standard Galvano-

ADDITIONAL ERRATUM.—PART II.

P. 695, line 8 from foot, *for* "equal" *read* "in most cases nearly equal"

For absolute

instrument it is necessary to know also the intensity, at the needle, of the magnetic field which exists independently of the current in the coil; since that with the field produced by the current gives the resultant-field in which the needle rests in equilibrium if subject only to magnetic action, or the magnetic couple system on the needle if besides magnetic forces, others (such as elastic forces) are effective in producing equilibrium.

A standard electrodynamometer is simply a standard galvanometer with the needle replaced by a movable

ERRATUM.

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CHAPTER VII

GALVANOMETRY AND MEASUREMENT OF CURRENTS

SECTION I

GALVANOMETRY

Standard Galvanometers and Electrodynamometers.

SINCE currents flowing in a given circuit are taken (p. 105 above) as proportional to the intensities of the magnetic fields they produce, and unit current is defined accordingly, the fundamental determinations of currents in absolute units must be made by some form of standard galvanometer, or standard electrodynamometer. The former is an instrument which exerts on a magnetic needle in any given position a couple which can be calculated with sufficient accuracy from the dimensions and arrangement of the coil-system, and the (approximately) known distribution of magnetism in the needle. For absolute measurements of currents by such an instrument it is necessary to know also the intensity, at the needle, of the magnetic field which exists independently of the current in the coil; since that with the field produced by the current gives the resultant-field in which the needle rests in equilibrium if subject only to magnetic action, or the magnetic couple system on the needle if besides magnetic forces, others (such as elastic forces) are effective in producing equilibrium.

A standard electrodynamometer is simply a standard galvanometer with the needle replaced by a movable

needle or coil-system, of such form and arrangement, and so suspended as to enable the system of couples acting upon it to be calculated for any position, or for a certain zero position, to which the movable coil-system is brought back by a proper displacement or distortion of the suspension or otherwise. In this case equilibrium is generally produced by means of a force due to elasticity or to gravity, which can be accurately determined.

The calculation of the magnetic forces has been given in Section I. of the last chapter for the most important arrangements of coils. We have only to consider the general construction and action of such instruments, the modes of suspension adopted for the needle or coil, the calculation or determination of the other than magnetic forces acting on the suspended system, and the practical operations of setting up and using the instruments.

Tangent and Sine Galvanometers. Dealing first with absolute galvanometers, we notice first that according to the mode in which they are used they are classed as *tangent galvanometers* or *sine galvanometers*. In the former the arrangement is such that the current flowing through the coils is (exactly or approximately) proportional to the *tangent* of the deflection of the needle from the undisturbed or initial position, in the latter the current is proportional to the sine of the deflection. We shall consider first the construction of galvanometers.

Formation of Coil Channel. As stated above, the standard galvanometer should be of such a form that the values of its indications can be easily calculated from the dimensions and number of

turns of wire in the coil. Such a galvanometer can be made by any experimenter who can turn, or can get turned, with accuracy a wooden or brass ring with a rectangular groove round its outer edge to receive the wire. It is indeed to be preferred that the experimenter should at least perform the winding of the coil and the adjustments of the needle, &c., himself, to make sure that errors in counting the number of turns, or in placing the needle at the centre of the coil are not made. If there are to be several layers of wire, the breadth and depth of this groove ought to be small in comparison with its radius, and each should be not greater than $\frac{1}{10}$ of the mean radius of the coil, which should be at least 15 cms.

Wire used for Coil. The gauge of the wire with which the coil is to be wound must depend of course on the purposes to which the instrument is to be applied, but it should be good well-insulated copper wire of high conductivity, and not so thin as to run any risk of being injured by the strongest currents likely to be sent through the instrument. For the exact graduation of current as well as of potential instruments, it is convenient to make it have two coils one of comparatively high, the other of low resistance. The latter may in some cases in which great accuracy is not required be a simple hoop of say 15 cms. radius, made of copper strip 1 cm. broad and 1 mm. thick. As however the distribution of the current in a massive conductor is uncertain in consequence of want of homogeneity in the material, and it is besides difficult to allow exactly for any irregularity that may exist where the ends are led out, and further, as it is difficult to make such a hoop of perfectly accurate shape, it is impossible to determine by calculation the exact constant of such a conductor, it is better to use instead several turns of thick wire. Each spire of the coil may then be regarded, as explained above, as a circular conductor coinciding with its circular axis.

Electrodes. To form electrodes to which wires can be attached the ends of the copper strip or thick wire are brought out side by side in the plane of the ring with sheet vulcanite or paper between for insulator. Insulated wires are soldered to the ends of the circle

thus arranged, and are twisted together for a sufficient distance to prevent any direct effect on the needle from being produced by a current flowing in them.

Winding of Coil. In constructing the fine wire coil the operator should first subject the wire to a moderate stretching force, and then carefully measure its electrical resistance and its length. He should then wind it on a moderately large bobbin and again measure its resistance. If the second measurement differs materially from the first, the wire is faulty and should be carefully examined. If no evident fault can be found, on the removal of which the discrepance disappears, the wire must be laid aside and another substituted. When the two measurements are found to agree the wire may then be wound on the coil. For this purpose the ring may either be turned slowly round in a lathe or on a spindle, so as to draw off the wire from the bobbin also mounted so as to be free to turn round The wire must be laid on evenly in layers in the groove (which may be done with the utmost uniformity with a self-feeding lathe) and the winding ended with the completion of a layer. Great care must be taken to count accurately the number of turns laid on. Error in counting may be avoided by following the plan used by Maxwell of winding a single layer of thin cord on a long wooden cylinder rigidly attached to the bobbin and therefore turning with it. A pin driven into the cylinder serves to indicate the end of one layer and the beginning of the next. After winding the resistance should be again measured, and if it agrees nearly with the former measurements the coil may be relied on.

Mounting of Coil. The ring carrying the coil thus made should then be fixed to a convenient stand in such a manner that if necessary it can be easily removed. The stand ought to be fitted with levelling screws, so that the plane of the coil may be made accurately vertical. A shallow horizontal box with a glass cover and mirror bottom may be carried by the stand near the level of its centre, and within this the needle and attached mirror or index suspended. Or, what is more convenient in many cases, a platform should be arranged below the level of the centre a sufficient distance to allow the magnetometer (such as one of those described in chapter II. above) to be placed with the centre of its needle at the level of the centre of the coil.

Needle and Suspension. The needle should be a single small magnet about a centimetre long, hung by a single fibre (half a cocoon thread) of unspun washed silk, at least 10 cms. long, or, better, by a fine quartz thread from the top of a tube fixed to the cover of the shallow box, or from the suspension head of the magnetometer.

if that is used, so that the centre of the needle when the coil is vertical is exactly the centre of the coil. To allow of the exact adjustment of the height of the needle, the fibre should be attached to the lower end of a small screw spindle, made so as to be raised or lowered, without being turned round, by a nut working round it above the cap of the tube.

Index or Pointer.

If the instrument is to be used with scale and pointer (or, as is desirable in some cases, is to be furnished with scale and pointer as well as mirror), the pointer may be made by drawing out a bit of thin glass tube at the blowpipe into a thread, so thick as to remain nearly straight under its own weight when suspended by its centre. In order that the zero position of the pointer may not be under the coil, the pointer ought to be fixed horizontally with its length at right angles to the needle, so as to project to an equal distance on both sides of it. To test that this adjustment is properly made, draw a couple of lines accurately at right angles to one another on a sheet of paper. Then suspend a long thin straight magnet over the paper, and bring one of the lines into accurate parallelism with it. Remove then the magnet and put in its place the little needle and attached index. If the index is parallel to the other line the adjustment has been correctly made. The needle may then be suspended in position, and the box within which it hangs closed to prevent disturbance from currents of air.

Adjustment.

Scale.

A circular scale graduated to degrees, with its centre just below the centre of the coil, and its plane horizontal is placed with its zero point on a line drawn on the mirror-bottom of the box at right angles to the plane of the coil, so that when the needle and coil are in the magnetic meridian the index may point to zero. The accuracy of the adjustment of the zero point is to be tested, as explained below, by finding whether the same current reversed produces equal deflections on the two sides of zero.

Adjustment of Scale.

To test whether the centre of this divided circle is accurately under the centre of the needle, supposed at the centre of the coil, draw from the point immediately under the centre of the needle two radial lines on the mirror-bottom, one on each side of the zero point and 45° from it, thus including between them an angle of 90°, and turn the needle round without giving it any motion of translation. If the index lies along these two radial lines when its point is at the corresponding division on the circle the adjustment is correct. Of course a fairly accurate first adjustment is previously made by placing the circle so that the two points each distant 45° from the zero lie on these straight lines.

Error from inaccurate centering can be almost completely

Avoidance of Error from Inaccurate Centering and Parallax.

eliminated by making the pointer extend across the circle and reading both ends of it.

When taking readings the observer places his eye so as to see the index just cover the image in the mirror-bottom of the box, and reads off the number of divisions and fractions of a division, indicated on the scale by the position of the index. Error from parallax is thus avoided.

Mirror and Scale.

A mirror rigidly attached to the needle may be used as in the magnetometer, instead of the needle and index, and observed by means of a telescope with attached scale, or, in the manner of an ordinary testing galvanometer, by means of a beam of light thrown by a lamp on the mirror and reflected to a scale.* Very conveniently a long fibre magnetometer carried on a platform fixed within the bobbin may be used for the needle and attached mirror. A hole, slot, and plane arrangement on the platform for the adjusted position will enable the magnetometer to be taken away and replaced at pleasure. The adjustments of scale, &c., are the same as those described in chapter II. above.

When a mirror is employed the coil is parallel to the undisturbed position of the needle (the magnetic meridian, when as usual the earth's field only is employed to give the return couple on the needle) when equal deflections on the two sides of zero are produced by reversing any current. The scales used should, if of paper, always be carefully glued to a wooden piece instead of being, as they frequently are, fixed with drawing-pins. Preferably however they should be ruled on glass by any one of the simple methods now available for copying an accurately engraved standard.

It is to be noticed that a mirror and straight scale placed at right angles to the undeflected position of the ray, and used in the ordinary way, give readings proportional to the tangents of double the angles of deflection.

Single Layer Tangent Galvanometer.

The author some time ago had constructed a standard galvanometer which seems to possess several advantages over the ordinary form. It consists of a cylindrical bobbin, about 50 cms. in diameter, and 25 cms. in length, wound with a single layer of fine wire. The needle (1 cm. long) is suspended at the centre of the bobbin, and the magnetic field produced by a current flowing in the wire is in this arrangement practically invariable over a distance in any direction at the centre considerably exceeding the length of the needle. Very accurate placing of the needle is thus not necessary, as a displacement of so much as half its length from the central position (an error

* See Vol. I. Chapter IV. p. 211, *et seq.*

of adjustment which is practically impossible with the slightest care) produces a quite imperceptible effect on the deflection with any given current.

The distribution of the wire, since there is only one layer, is known with perfect certainty, and hence the constant of the instrument can be calculated with great exactness. At each end of the bobbin is wound one of two equal coils of small transverse dimensions in comparison with their radii. These are of thick copper wire arranged so as to form a Helmholtz double coil galvanometer of the kind described above (p. 254), available for strong currents.

When the instrument was being designed it was thought desirable to have the bobbin made of some material which could not contain magnetic substances, in sufficient quantity to affect the accuracy of measurements of currents flowing in the wire. The fear then felt by the author that the bobbins of brass ordinarily employed for standard galvanometers might very probably contain iron, in sufficient quantity to cause disturbance through its induced magnetization, has since been found by Prof. T. Gray to be, in part at least, justified. The measurement of currents made by a new standard galvanometer were found by him to be so much disturbed by the effect of magnetic substances contained in the walls of a brass box surrounding the needle as to be practically useless.

Built up Bobbin of Wood.

It was resolved therefore to construct a bobbin of wood in such a manner as to avoid risk of serious alteration of figure by warping, or of dimensions through variation in the amount of moisture contained in the wood. A large number of pieces of mahogany were cut from a dry well-seasoned board about ½ inch thick. Each piece was about 4 cms. broad, 20 cms. in length, and was cut so as to form a segment of a ring the outside diameter of which was about 50 cms. and the inner diameter about 8 cms. less. Four of these cut so that the grain of the wood ran in different directions in adjoining pieces and placed end to end gave a complete circular ring, or rather cylinder, ½ inch in length. Above that was placed a similar ring with the grain of the wood in the pieces crossing that in the pieces below, and the pieces themselves overlapping the end joints in the preceding ring. Above that was placed another ring, and so on until the whole bobbin, rather more than 25 cms. in length, had been built up. The cylinder thus roughly formed was then turned carefully down to cylindrical figure of the size desired, and as nearly truly circular as possible, and the pores all over the surface, inside and outside, filled with spirit varnish to prevent the absorption of moisture.

[A bobbin thus built up of pieces of wood will probably not take or keep so true a figure as one made of metal, but there can be no doubt of its great superiority over the ordinary bobbin of wood, made out of one piece. For all except purposes for which the highest accuracy is required, it may be relied on to give correct results. If a metal bobbin is preferred the material ought to be carefully tested in a magnetic field, and rejected if appreciable induced magnetization is detected.]

Two edges of wood, projecting slightly beyond the outside cylindrical surface, were fixed at the ends to keep the wire in its place. The coil was then carefully wound, the turns counted, and the wire covered with "American cloth" to preserve it from injury. The two ends of the thin wire coil were brought out together at one end of the coil for connection to two electrodes closely twisted together and several yards in length, by which the instrument could be joined to any circuit in which it might be required. That end of the wire which had to be carried from the further extremity of the coil was (supposing the coil set up in position) brought along horizontally in a vertical plane through the axis of the coil until it met the other extremity at the termination of the last spire of the coil. The current in this part of the wire of course just compensates by its effect on the needle that of the component of current in each element of the spires in the direction of the axis.

Tangent Galvanometer Principal Constant of Single Layer Bobbin.

The couple given by (57) of last chapter is, if as a first and usually sufficient approximation the first term of the expression only is taken, $2\pi N\gamma M \cos\theta\,(a^2+b^2)^{\frac{1}{2}}$, where M is the magnetic moment of the needle, N the total number of turns in the coil, a the radius of the coil, b its half length, and θ the angle which the needle makes with the mean plane of the coil. The return couple given by the permanent magnetic field (horizontal intensity H) is $MH\sin\theta$, if the mean plane of the coil and the axis of the needle are made to coincide when the deflection is zero, by the adjustment explained below. Thus we have equating these couples

$$\gamma = \frac{(a^2+b^2)^{\frac{1}{2}}}{2\pi N} H \tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For the thick wire coils the deflecting couple Θ is given by (24) of last chapter, and for equilibrium we have $\Theta = MH\sin\theta$. If we put $\Theta - \gamma MG\cos\theta$, we get

$$\gamma = \frac{H}{G}\tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where G is the quantity obtained by dividing the expression in the sign of (24) by $M\gamma$. G is sometimes called the galvanometer constant.

Sine Galvanometer.

In a sine galvanometer the coils are made movable round a vertical axis through the centre of the needle, and when the needle is deflected the coils are turned until an equilibrium position is obtained in which the needle and mean plane of the coils are again parallel. Thus $\cos\theta$ in the expression for Θ given in last chapter must be put equal to unity. The deflection θ of the needle is equal to the angle through which the coils have been turned, and is usually measured by observing this angle by means of a finely divided scale provided with verniers and reading microscopes. For such an instrument we have instead of (2)

$$\gamma = \frac{H}{G}\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the values of G for the different types of instrument given by the various expressions contained in Chapter VI., the inclination of the needle to the plane of the coil is of course to be put equal to zero.

Fitzgerald's Standard Galvanometer.

An instrument capable of being used at pleasure either as a tangent or sine galvanometer has been designed by Professor G. F. Fitzgerald, and is shown in Fig. 74. Its distinctive peculiarities consist in an arrangement of coils which permits the constant of the instrument to be determined with the coils in position, and a very ingenious arrangement for measuring the deflections of the needle and the coils from the adjusted position for no current.

Arrangement of Coils.

The coils are visible through a plate-glass casing and can be measured *in situ*. The deflection of the needle is observed in the following manner on the cylindrical scale shown in the figure. A pair of small totally reflecting prisms, with their reflecting surfaces inclined at 45° to the horizontal, are carried by the magnet, and give images of diametrically opposite parts of this scale, and show on these images of one and the same line or mark. These are seen at the same time in the field of view of a microscope which receives the light from the mirrors. Thus the arrangement is equivalent to, but much more sensitive than, a pointer playing round a graduated circle and read at both ends to eliminate error from inaccuracy of centering.

The coils can be turned round to follow the magnet, and their position observed on the same cylindrical scale: so that a single scale serves for the use of the instrument both as a tangent galvanometer and as a sine galvanometer.

It has been noticed above that the ordinary method of using

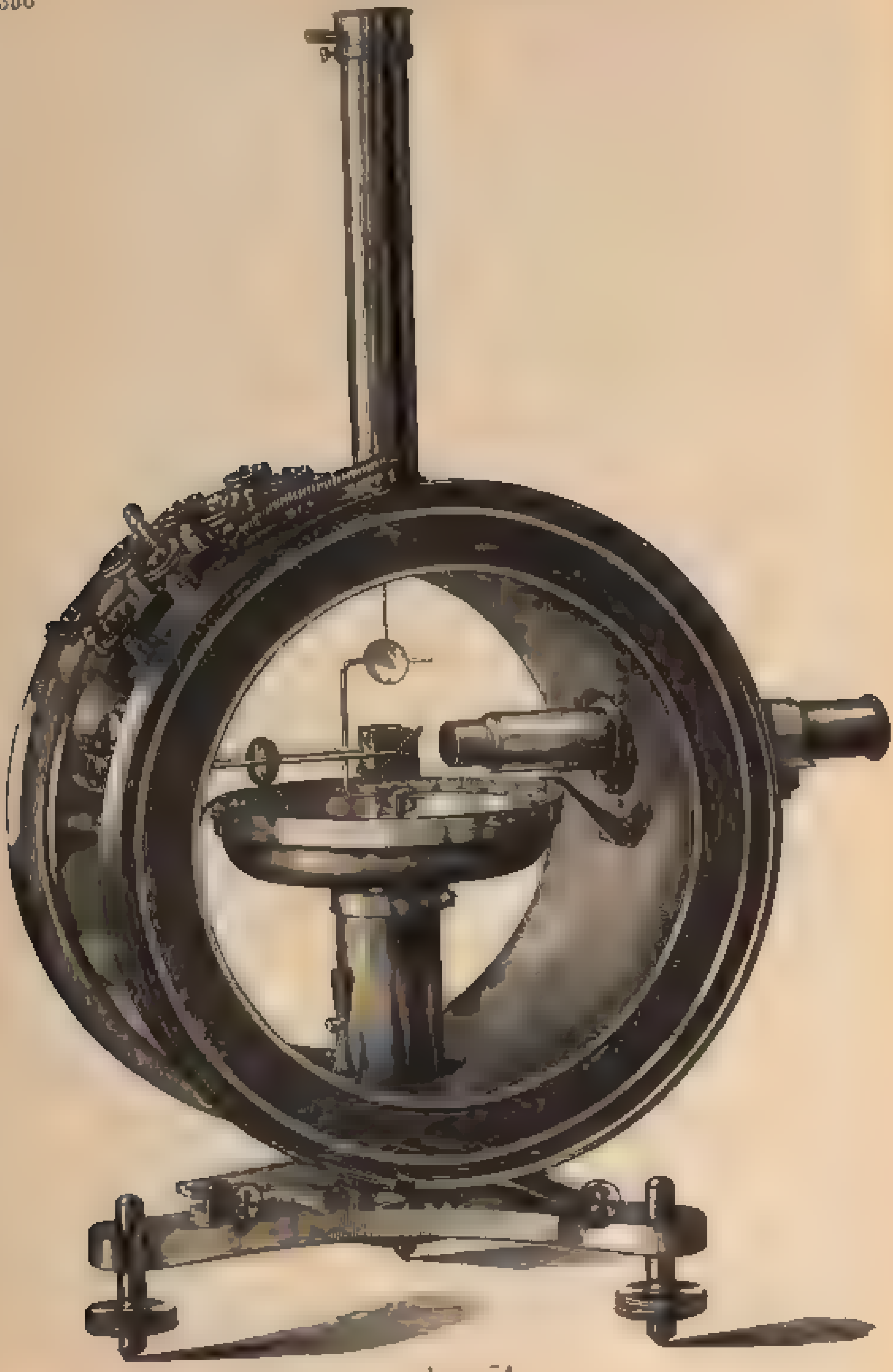

FIG. 74.

the mirror and scale gives with a straight scale properly adjusted the tangent of twice the angle of deflection. In Professor Fitzgerald's instrument besides the arrangement just described for reading the deflection a mirror is provided attached at 45° to the axis of suspension. A vertical ray of light falling upon this mirror is sent out horizontally through one of the plate-glass sides of the case to a horizontal scale. As the mirror turns round the plane of reflection turns with it, and through the same angle, so that with a straight scale placed at right angles to the undisturbed position of the ray, the readings on the scale are proportional to the tangents of the actual deflections.

Method of Reading Deflection.

Fig. 75 shows a sine galvanometer designed by Prof. T. Gray. A single layer of wire is wound on a tube of about 10 cms. diameter, and at least ten diameters in length. If the

T. Gray's Sine Galvanometer.

Fig. 75.

coil be uniformly wound with n turns per unit of length, and l be its half-length and a its radius, the force f per unit of current at the centre is (see p. 261 above) $4\pi nl\,(a^2 + l^2)^{-\frac{1}{2}}$. This becomes $4\pi n$ if l be great in comparison with a, for example if l is ten times a, the value of f is only $\frac{1}{2}$ per cent. less than $4\pi n$, as is shown by the equation

$$f = 4\pi n\left(1 - \frac{1}{2}\frac{a^2}{l^2} + \frac{3}{8}\frac{a^4}{l^4} - \dots\right)$$

$$= 4\pi n\left(1 - \frac{1}{200} + \frac{3}{80000} - \dots\right)$$

$$l = 10a.$$

Thus the very exact determination of the radius is not a matter of very great importance, and if the coil be very uniformly wound over the middle part, and very fairly regularly elsewhere, the value of f will be given with great accuracy by the

first two terms of the series. The uniformity of the winding can be made almost quite perfect by laying on the wire under a moderate tension by means of a self feeding lathe.

Arrangement of Coil

The coil is wound on the tube T (Fig. 75). The ends of the wire are attached to pins p_1, p_2, and a wire w running parallel to the axis of the coil connects p_2 to a third pin p_3 close to p_1. A pair of flexible electrodes well twisted together connects $p_1 p_3$ to a pair of terminals on the platform P. The tube is mounted, as shown, on the circular platform P which is furnished with levelling screws L, L, L, and can be turned round the vertical axis V, the supports f, f sliding on the platform and maintaining the tube in a horizontal position. The scale S on the edge of the platform enables the angle through which the coil is turned to be measured.

Method of Reading Deflection

The needle is suspended at the centre of the tube, and may be either a light polished steel disk, or a plane or concave mirror with attached steel magnets. The arrangement preferred is as follows:—At one end of the tube is a short scale s facing towards the mirror (which is plane) and illuminated by light entering a small hole at that end of the tube, and thrown on the scale by a reflecting prism or inclined mirror. At the same end of the tube is a fixed mirror M, also turned towards the suspended mirror m. By means of the telescope t at the other end of the tube, fixed above the centre with its vertical cross-wire as nearly as may be in the medial vertical plane of the coil, the scale s is seen by light which has suffered two reflections, one at m the other at M, and thus the angle through which the needle has been turned can be obtained.

For the scale s may be substituted a narrow slit, or, preferably, a wide slit, or hole, crossed by a wire, in front of which within the tube is fixed a lens, and for the telescope a sheet of obscure glass. An image of the slit or wire is focused by the lens on the obscure glass, and the position of this can be read from without on a scale fixed to or engraved on the glass.

Or, the plane mirror m may be replaced by a concave spherical mirror as in an ordinary Thomson's galvanometer, and the obscure glass carried by a sliding tube which can be pushed out or in to give a sharp image of the slit or wire.

Method of using Instrument.

The method of using the instrument is as follows: It is placed in a well-lighted room, and the platform P is levelled by means of the screws L. The coil is then turned until the central division of the scale s coincides with the cross-wire of the telescope (or the zero of the scale on the obscured glass), and the reading on the scale S is taken. Then a steady current is passed through the coil, and the angle noted through which

the tube has to be turned to bring the central division of s again to the cross-wire of the telescope. The current is then reversed, and the scale s moved if necessary until the angles on the two sides of zero are equal. If θ is this deflection on the scale S the current is given by the equation

$$\gamma = \frac{H \sin \theta}{4\pi n\left(1 - \frac{1}{2}\frac{a^2}{l^2}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The angle θ can evidently be attained with great accuracy by very accurate division of the scale S, and reading it with a vernier and microscope.

We now discuss shortly some general propositions regarding the action of galvanometers, their adjustment and sensibility.

Theory of Tangent Galvanometer

We shall suppose to begin with that the forces acting are wholly magnetic, and that the suspension is such as to prevent other than horizontal forces from affecting the needle. When no current is flowing the needle rests horizontal with its axis parallel to the permanent magnetic field, or to its horizontal component. The needle will take up a new position making an angle θ with the plane of the coil. The angle which the needle now makes with its initial position is $\theta - a$, say. The couple, Θ, acting upon the needle is given by equations (13), (20), (21), &c. of last chapter. If M be the magnetic moment of the needle, and H the horizontal component force of the permanent field, we have for the return couple $MH \sin (\theta - a)$. Hence

$$\Theta = MH \sin (\theta - a).$$

But we may write $\Theta = \gamma MG \cos \theta$, and therefore

$$\gamma = \frac{H}{G} \frac{\sin (\theta - a)}{\cos \theta} \quad . \quad . \quad . \quad . \quad (5)$$

G, as shown by (13) above, in general depends on θ. If the needle however be sufficiently short the terms depending on θ disappear. G is then what is called the *galvanometer constant.*

If a is zero (5) becomes

$$\gamma = \frac{H}{G} \tan \theta \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and if G is independent of θ the current is proportional to the tangent of the deflection. Hence the name of the instrument.

Adjustment of Instrument.

The instrument is generally set up so that a is zero or very nearly so. This adjustment may be made as follows. Supposing the stand of the coils fitted with a level by means of which the coils can be placed in a vertical position, the instrument is thus levelled and placed by guess with the mean plane of the coils as nearly as may be parallel to the needle. The coil is then joined up with a voltaic cell and reversing key so that a current can be sent in either direction through it. A current is sent through the coils, and the deflection θ of the needle is observed by means of the mirror or pointer attached to the needle. The current is then reversed and the opposite deflection observed. If this is the same as before the coil is properly placed. If not let the numerical value of the first deflection without regard to sign be θ, and of the second θ', and let a be the (unknown) angle which the mean plane of the coils makes with the needle. Supposing G the same in both cases, which it will approximately be if θ is nearly the same as θ', we have by (5)

$$\frac{\sin(\theta - a)}{\cos\theta} = \frac{\sin(\theta' + a)}{\cos\theta'}.$$

This gives

$$\tan a = \frac{\sin(\theta' + \theta)}{\cos\theta'(\cos\theta' - \cos\theta)}$$

which shows that if $\theta > \theta'$ the coil is turned through an angle a, in the direction of the first deflection; if $\theta < \theta'$ the coil deviates from the position of the needle by an angle a in the direction of the second deflection.

The actual value of a can thus be calculated, and if the coils can be turned through any required angle the correction of position can at once be made. If, however, there is no provision for turning the coils through a definite angle, the correction must be made by guess from the direction of the greater deflection, then the new position tested, and if necessary corrected, and so on.

The galvanometer is sometimes set so that $a = 45°$, and the current then made to flow so that the deflection is towards the coil. Then by (1) (changing the sign of the right-hand side to keep γ positive) Coil at 45° to Meridian.

$$\gamma = \frac{H}{G}\frac{\sin\left(\frac{\pi}{4} - \theta\right)}{\cos\theta} = \frac{\sqrt{2}}{2}\frac{H}{G}(1 - \tan\theta) \quad . \quad (6)$$

It is to be noticed that here θ is to be taken positive when it is on the same side of the coil as the initial position of the needle, and negative when it is on the opposite side. The deflection of the needle may thus be as great as 90° from the initial position. For this value of the deflection the current is $\sqrt{2}\,H/G$.

The adjustment to this position may be made by first placing the galvanometer as described above so that its mean plane is parallel to the undisturbed position of the needle, and then turning the instrument round through exactly 45°. This mode of using the instrument, though it gives a wider range, is attended with the inconvenience that the deflection if considerable can only be taken in one direction.

Sensibility of Galvanometer.

The sensibility of a galvanometer may be defined as the reciprocal of the current required to produce a definite small angular deflection of the needle, or, which comes to the same thing, it may be taken as measured by the angular deflection produced by a specified current, for example, a micro-ampere (one millionth of an ampere) Frequently if the galvanometer be a reflecting one it is regarded as inversely proportional to the current required to produce a deflection of one division of the scale, but this of course is a function of the arrangement of mirror and scale, and not merely of the coil.

Measurement of Sensibility.

The sensibility can be determined by sending through the coil, arranged as will generally be necessary, with some considerable resistance in circuit, and shunted, if need be, by a resistance the ratio of which to the resistance of the coil is known, a current from a cell of known electromotive force, calculating the current, and observing the deflection.

The actual merit of the instrument cannot however be completely determined by such a process, as that depends on length of period of the needle, steadiness of zero, &c., which are not here taken account of. For

an elaborate comparison of sensibilities of galvanometers see a paper by Messrs Ayrton, Mather, and Sumpner, *Phil. Mag.*, July 1890.

Sensibility for Different Positions of Needle.

The sensibility of a galvanometer, for different positions of the needle, is the ratio of the increase of deflection to the increase of the current, or $d\theta/d\gamma$. This is a maximum in the case of a tangent galvanometer for zero deflection.

When however the deflection is 45° a given percentage of increase or diminution of the current produces a maximum increase or diminution of deflection, that is to say $\delta\theta/(\delta\gamma/\gamma)$ is then a maximum; and hence the instrument is sometimes erroneously stated to be most sensitive when the deflection is 45°. The only importance in making the deflection 45° lies in the fact that with this deflection a given small error in reading the angle will have a minimum effect on the estimation of the current.

To prove these propositions we observe first that by (2)

$$\frac{d\theta}{d\gamma} = \frac{G}{H}\frac{1}{1 + \tan^2\theta},$$

and this is obviously a maximum when $\theta = 0$.

Again let the reading be in error $\delta\theta$ when the deflection is really θ. Then the current is estimated by (2), and if γ is the true current the estimated current is $\gamma \pm \delta\gamma$, or $\gamma \pm d\gamma/d\theta.\delta\theta$. The error in estimation of the current is $\delta\gamma/\gamma$ or $d\gamma/d\theta.\delta\theta/\gamma$. But

$$\frac{1}{\gamma}\frac{d\gamma}{d\theta}\delta\theta = \frac{1 + \tan^2\theta}{\tan\theta}\,\delta\theta.$$

This is a minimum when $(1 + \tan^2\theta)/\tan\theta$ is a minimum, that is when $\tan\theta = 1$, or $\theta = 45°$.

Torsion of Suspension Fibre.

In every properly constructed absolute galvanometer the torsion of the suspension ought to be negligible, and if a quartz thread, or a sufficient length of silk fibre be used, it will be negli-

gible. The amount of torsion may however be estimated as follows. Let the needle supposed initially in the magnetic meridian be turned once or more times completely round, and let its deviation from the magnetic meridian in its new position of equilibrium be noted by means of index and divided scale, or mirror and scale or telescope provided for the purpose. If α be the angular deflection of the magnet from the magnetic meridian produced by turning the magnet once round, the angle through which the thread has been turned is $2\pi - \alpha$. The couple produced by this torsion has for moment $MH \sin \alpha$. Hence by Coulomb's law of the proportionality of the force of torsion to the twist given, the couple corresponding to deflection θ is $MH \sin \alpha \, \theta/(2\pi - \alpha)$. Thus if a current γ produces the deflection θ the equation of equilibrium is

$$\gamma G \cos \theta = H \left(\sin \theta + \frac{\theta}{2\pi - \alpha} \sin \alpha\right),$$

and therefore

$$\gamma = \left(1 + \frac{\theta}{2\pi - \alpha} \frac{\sin \alpha}{\sin \theta}\right) \frac{H}{G} \tan \theta . \qquad (7)$$

Electrodynamometers.

We now consider absolute electrodynamometers. The first instrument of this kind seems to have been invented by W. Weber, and used by him in his researches on the mutual action of currents. Electrodynamometers have advantages over galvanometers (1) in having no magnet, and therefore avoiding altogether uncertainty as to distribution of magnetism; (2) in not involving for the reduction of their indications any knowledge of the intensity of the earth's field; but are inferior in point of sensibility, and as the return couple is generally given by a bifilar or torsion suspension the accurate estimation of its value is a matter of some difficulty.

The galvanometer designed by Professor Fitzgerald and described above could, as he has pointed out, easily be adapted for use as an electrodynamometer. All that is required is the substitution of a proper suspended coil,

and a bifilar suspension for the needle. The same arrangement of mirrors and cylindrical scale would be available to give the deflections.

B. A. Committee's Electro-dynamometer.

We shall describe the general arrangement and mode of using an electrodynamometer with reference to the instrument made by Mr. Latimer Clark for the British Association Committee on Electrical Standards, and illustrated in Figs. 76, 77.

The first of these figures shows the general arrangement of the instrument, the second the details of the suspension.

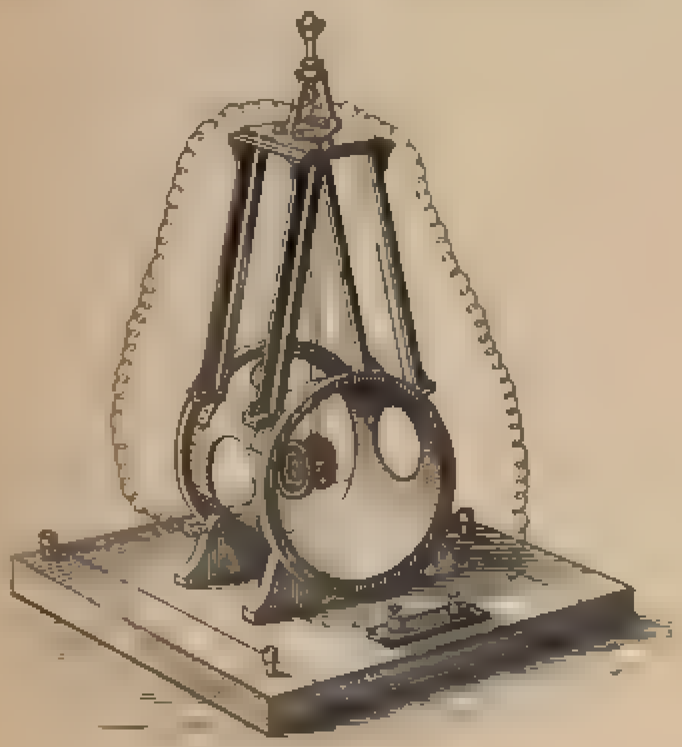

FIG. 76.

The bifilar consists of two wires the tension of which is maintained the same by their being attached to a piece of silk thread which passes over a pulley, as shown in Fig. 77. The distance between the threads is adjusted by two guide pulleys which can be set at any required distance apart. The current is led into the suspended coil by means of the suspension wires. Arrangements are also made whereby the current can be sent in either direction through each coil.

Theory of Instrument.

The instrument as stated above has both its stationary and movable coil systems, constructed on Helmholtz's plan of two equal parallel coils at a distance apart equal to their radius. The suspended coil system is hung so that it is concentric with the fixed coils, and when there is zero deflection their planes are at right angles to one another.

Fig. 77.

When the axis of the suspended coil makes an angle $\pi/2 - \phi$ with the plane of the fixed coil, the couple Θ due to the currents and *tending to increase* the deflection, θ, has the expression given in (45) or (46), page 271 above, with sign changed and $\cos(\pi/2 - \phi)$ substituted for $\sin\phi$. Again the suspended coil is acted on by a couple due to the earth's magnetic force H, and tending to diminish $\pi/2 - \phi$. Thus (46) gives for the former couple $4N n\gamma\gamma' G_1 g_1 \cos(\pi/2 - \phi)$, since $_0Z'_1 = 1$; and for the other couple $2n\gamma' g_1 H \sin\theta'$, where N, n, γ, γ', are the numbers of turns and the currents in the fixed and movable coils respectively, and θ' is the angle which the axis of the movable coil makes with the magnetic meridian. Thus if L be the return couple due to the suspension, and the plane of the fixed coil make an angle α with the magnetic meridian, and an angle β with the axis of the movable coil in the undisturbed position, we have for equilibrium

$$4N n\gamma\gamma' G_1 g_1 \cos(\theta + \beta) - 2n\gamma' g_1 H \sin(\theta + \beta + \alpha) - L = 0.$$

The value of L, if θ be small, is proportional to $\sin\theta$, so that $L = F\sin\theta$.

$$F\tan\theta = 4Nn\gamma\gamma' G_1 g_1 (\cos\beta - \tan\theta\sin\beta) \\ -2n\gamma' g_1 H\{\tan\theta\cos(a+\beta) + \sin(a+\beta)\}$$

and if a and β be both small

$$\tan\theta = \frac{1}{F}\{4Nn\gamma\gamma' G_1 g_1 \cos\beta - 2n\gamma' g_1 H\sin(a+\beta) \\ - \frac{1}{F^2}(16N^2n^2\gamma^2\gamma'^2 G_1^2 g_1^2 \sin\beta + 8^3 NnH\gamma\gamma'^2 G_1 g_1^2)\} \quad . \quad (8)$$

Methods of Using Instrument.

Now a direction of the current in the coils being assumed as positive, the currents are sent through the two coils according to the adjoining scheme and produce the corresponding deflections $\theta_1, \theta_2, \theta_3, \theta_4$.

	γ	γ'
θ_1	+	+
θ_2	−	−
θ_3	+	−
θ_4	−	+

Thus we get by substitution in (8) and reduction

$$\gamma\gamma' = \tfrac{1}{4}\,\frac{F}{4NnG_1g_1\cos\beta}(\tan\theta_1 + \tan\theta_2 - \tan\theta_3 - \tan\theta_4) \quad . \quad (9)$$

If $\gamma = \gamma'$ this gives the value of γ^2.

By this method H is eliminated, and it is the best method to adopt when readings have to be obtained quickly, as when the current is varying. If however the current is constant enough

the head of the bifilar suspension may be turned round until the suspended coil is brought back to its original position after deflection. When this is the case the angle θ through which the coil is deflected from its equilibrium position is clearly equal and opposite to the angle β, through which the head of the bifilar has been turned round from the position of parallelism with the plane of the coil. We have thus $\theta = -\beta$. For equilibrium we have the equation

$$F \sin \beta = -4Nn\gamma\gamma' G_1 g_1 + 2n\gamma' g_1 H \sin \alpha.$$

Taking four deflections according to the above scheme, we get four readings of the head of the bifilar $\beta_1, \beta_2, \beta_3, \beta_4 = -\theta_1, -\theta_2, -\theta_3, -\theta_4$, and so

$$F \sin \beta_1 = -F \sin \beta_3 = -4Nn\gamma\gamma' G_1 g_1 + 2n\gamma' g_1 H \sin \alpha.$$

$$F \sin \beta_2 = -F \sin \beta_4 = -4Nn\gamma\gamma' G_1 g_1 - 2n\gamma' g_1 H \sin \alpha$$

Hence

$$\gamma\gamma' = \frac{F}{4NnG_1 g_1} (\sin \beta_1 + \sin \beta_2 - \sin \beta_3 - \sin \beta_4) \quad . \quad (10)$$

in which again H does not appear.

Dynamometer made of Two Single Layer Coils.

An absolute electrodynamometer may be constructed, as described above (p. 274), of two single-layer coils placed with their centres in coincidence. If the ratio of length to radius be as proposed above in each case $\sqrt{3} : 1$, the value of the couple due to the action of the currents will be as given in (56), p. 276, $8\pi^2 nn'\gamma\gamma' a^2 x\xi / \sqrt{a^2 + x^2} \cdot \cos(\pi/2 - \phi)$, where n, n' are the numbers of turns per unit length in the two coils, x, ξ, a, a, their respective half-lengths and radii, γ, γ', the currents in them, and $\pi/2 - \phi$ the angle which the axis of the movable coil makes with the mean plane of the fixed coils. This with $\pi/2 - \phi$ replaced by $\theta + \beta$ is to be used in the formulae (9) and (10) given above, instead of $4Nn\gamma\gamma' G_1 g_1 \cos(\theta + \beta)$. Thus the equations replacing (9) (10) for this case are

$$\gamma\gamma' = \tfrac{1}{4} \frac{F\sqrt{a^2 + x^2}}{8\pi^2 nn' a^2 x\xi} (\tan \theta_1 + \tan \theta_2 - \tan \theta_3 - \tan \theta_4) \quad . \quad (11)$$

$$\gamma\gamma' = -\tfrac{1}{4} \frac{F\sqrt{a^2 + x^2}}{8\pi^2 nn' a^2 x\xi} (\sin \beta_1 + \sin \beta_2 - \sin \beta_3 - \sin \beta_4) \quad . \quad (12)$$

Non-Absolute Instruments.

Galvanometers and electrodynamometers are very frequently used which by themselves are not capable of giving measurements of currents in absolute units. Such instruments are "calibrated" by some reliable method, so that the absolute values of the currents corresponding to any given deflections are known. In general they differ very much from the so-called absolute instruments in the arrangement of their coils, &c., which has had chiefly in view the attainment of the greatest possible sensibility.

We shall distinguish between instruments which have in their coils a great many turns of fine wire, so that the resistance of the coil system amounts to at least several hundred ohms, and those instruments the resistance of which is comparatively low. The former are very frequently called "potential" instruments or voltmeters from their use in determining the difference of potential between two points in a circuit at which the terminals are applied; the latter are called low resistance or "short coil" instruments, and sometimes (when their resistances are so low that one of them can be placed in series with the working circuit without materially increasing its resistance) "current" or amperemeters.

First taking galvanometers, we shall establish some general theorems regarding the arrangement of their coils, then very shortly discuss their graduation for absolute measurements, and finally deal with graduated electrodynamometers.

Proper Gauge of Wire for Given Bobbin and Generator.

In the first place, let the galvanometer have a certain cylindric channel which is to be filled with wire, and let it be required to find the gauge of wire with which it ought to be wound if it is to be used in circuit with an electrical generator of given electromotive force and resistance. Let a be the radius of cross-section of the wire employed, c the thickness of the covering, and S the cross-section of the channel made by a plane through the axis. The portion of the cross-section occupied by each turn will be $(2a + 2c)^2$ if the turns are arranged in square order in the cross-section, and $(2a + 2c)^2 \sqrt{3}/4$ if they are arranged in triangular order. This includes the space occupied by the covering and the vacant spaces between the spires.

Considering at present the first case only we see that the number of turns is $S/(2a + 2c)^2$, if any inaccuracy introduced by its being impossible to fit an exact number of turns into a complete layer is neglected. If r be the mean radius of the cross-section of the channel, the whole length of wire is approximately $2\pi r S/(2a + 2c)^2$. But ρ denoting the specific resistance of the wire, the resistance per unit length is $\rho/\pi a^2$, and the whole resistance R of the coil is $\frac{1}{2}\rho r S/(a + c)^2 a^2$. For a given current

the magnetic force at the needle is proportional to the number of turns, and the magnetic force parallel to the axis may therefore be written $\frac{1}{2} AS\gamma (a + c)^2$ where A is a constant. If E be the electromotive force of the generator, and R' the resistance of the generator and wires connecting it to the galvanometer bobbin, we have

$$\gamma = \frac{E}{\dfrac{\rho r S}{2a^2 (a + c)^2} + R'}$$

and for the axial component of magnetic force

$$F = \frac{ASE}{\dfrac{\rho r S}{2a^2} + R'(a + c)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Since the numerator is constant, this has its maximum value when the denominator is a minimum. Calculating in the usual manner the necessary condition, we find the equation

$$a^4 + ca^3 = \frac{\rho r S}{2R'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

a biquadratic for the determination of the corresponding value of a. But for the reciprocal $1/R$ of the resistance of the bobbin we have the value $2(a + c)^2 a^2/\rho r S$, and this used with the last equation gives

$$\frac{R}{R'} = \frac{a}{a + c}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

or the resistance of the bobbin should have to the resistance of the generator and connecting wires the ratio of the radius of the wire when bare to its radius when covered.

If the spires are arranged in triangular order, the equation of condition corresponding to (4) is

$$a^4 + ca^3 = \frac{2\rho r S}{\sqrt{3} R'},$$

and since, in this case, $1/R = \sqrt{3} a^2 (a + c)^2/2\rho r S$, we have the same result as before.

It may be remarked here that the magnetic effects of a given bobbin wound with wire of different gauges, the thickness of coating in which bears a constant ratio to the diameter of the wire, and traversed in each case by the same current, are proportional to the square root of the resistance of the coil. For we have then $(a+c)/a=k$, or $a+c=ka$. Thus by what has been shown at pp. 369, 370, the magnetic effect is proportional to $1/k^2a^2$, and the resistance to $1/k^4a^4$; hence the magnetic action varies as $\sqrt{R}$. Magnetic Action of Bobbin as depending on Gauge of Wire.

It is obvious that this is also true when the thickness of the covering is so small as to be negligible.

The best shape of cross-section for the bobbin of an ordinary galvanometer is shown in Fig. 78. The curve forming the external boundary of the cross-section is given by the equation, Best Shape of Section of Bobbin.

$$r^2 = p^2 \sin \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where r is the distance of any point P of the surface from O the centre of the coil, θ the angle POM which OP makes with the axis OM, and p a constant.

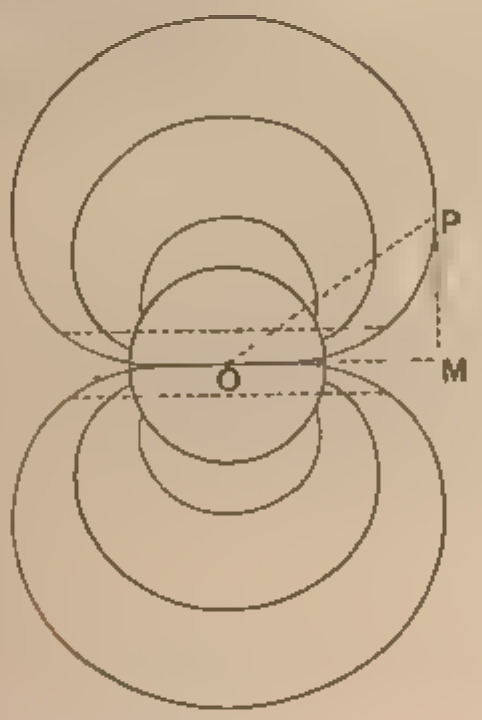

FIG. 78.

To prove this note that the axial magnetic force due to a single turn of wire of radius a, is proportional to a/r^3, that is to $\sin \theta/r^2$. Let now this turn be transferred to any point outside the surface, fulfilling this equation, on which it lies. Then

whatever the radius of the circle into which it is now bent, the length of arc which it furnishes is the same as before, and so the axial magnetic force is proportional to the new value of $\sin\theta/r^2$. But for every point of the cross-section outside the boundary fulfilling (16) the value of $\sin\theta/r^2$ is smaller, and for every point within the boundary is greater, than for a point on the surface. Thus a given length of wire produces a greater or less axial magnetic force according as it is wound without or within this surface. If then a coil be wound of any shape of cross-section the external boundary of which does not fulfil (16), by removing the wire from one part of the coil to another, the cross-section may be brought to this shape, and the axial magnetic force increased.

Fig. 78 shows curves for different values of p, and the two parallel dotted lines indicate a cylindrical chamber left for the needle.

Effect of Grading the Gauge of Wire in Bobbin.

In the investigation given above (p. 370) of the best gauge of wire with which to fill a given channel, when the bobbin is to be used with a generator of known electromotive force, it has been assumed that the wire must be of uniform thickness; and we have just seen what is the best form of cross-section to give a coil which is to contain a given volume of wire. When a coil is wound, however, each additional turn of wire, though it increases the axial magnetic force for a given current, also increases the resistance in circuit, and thereby diminishes the current produced by a given electromotive force. We shall now inquire whether by winding the outer layers of thicker wire the effect of increased resistance can be reduced to a minimum.

The volume of the coil supposed without chamber for the needle is

$$2\pi y \times \tfrac{1}{3}\int_0^\pi r^3 d\theta$$

where y is the distance of the mean point of the cross section from the axis. Now

$$y = \frac{\iint r^2 \sin\theta\, dr\, d\theta}{\tfrac{1}{3}\int r^3 d\theta}$$

the limits of integration being 0 and $p\,(\sin\theta)^{\frac{1}{2}}$ for r, and 0 and π for θ. Hence on the supposition already made

Theory of Graded Coil.

$$\text{volume of coil} = \tfrac{2}{3}\pi p^3 \int \sin^{\frac{5}{2}}\theta \, d\theta$$

$$= \tfrac{1}{3} N p^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

if $N = 2\pi \int_0^{\pi} \sin^{\frac{5}{2}}\theta \, . \, d\theta$, which does not depend on the dimensions or shape of the coil. The chamber containing the needle should be made as small as possible,* as the part of the coil immediately surrounding the magnet is the most valuable; but it will always cut away a part of the coil depending on p, which may be denoted by $f(p)$. The actual volume of the coil is thus $\frac{1}{3} N p^3 - f(p)$.

If now dl be an element of length of the wire composing the coil, and p the parameter of the generating curve of the surface on which it lies, then since $1/p^2 = \sin\theta/r^2$, the axial magnetic force at the centre is $\gamma \int dl/p^2$ ($= \gamma G$, say), where p is a function of the whole length, l, of wire in the coil from some chosen point, say the inner end, to dl. We shall suppose the wire to be of a different gauge at different places in the coil. If its radius at dl be a, the thickness of the covering there c, and the winding be in square order, the volume occupied by dl is $dl \, . \, (2a + 2c)^2$, so that the whole volume is $\int dl \, . \, 4(a + c)^2$ where a (and c if not constant) is a function of l, and the integral is taken throughout the whole length of wire in the coil.

Let the coil be considered as made up of layers each fulfilling the equation $r^2 = p^2 \sin\theta$, but each for its own value of p, so that a is a function of p. We have thus for the volume of the space between the layers corresponding to p and $p + dp$ the expression $Np^2 dp - f'(p)dp = (2a + 2c)^2 dl$, if dl be now put for the length of wire in this space. Thus $dl = \{Np^2 dp - f'(p)dp\}/(2a + 2c)^2$, and we get

$$dG = \frac{Np^2 - f'(p)}{p^2 4(a + c)^2} \, dp \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$dR = \frac{\rho}{\pi a^2} \, \frac{Np^2 - f'(p)}{4(a + c)^2} \, dp \quad . \quad . \quad . \quad . \quad . \quad (19)$$

If the generator have as before an electromotive force E, and R' denote as before the resistance of the generator and connecting

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* For the manner of winding the space close to the magnet see p. 376 below.

wires, we have $\gamma = E/(R + R')$, and $\gamma G = EG/(R + R')$. To make γG or $G/(R + R')$ a maximum by properly grading the wire, we have so to choose the diameter for each layer that the contribution of the layer to $G/(R + R')$ shall be as great as possible. Now imagine any layer to be taken away from the coil, everything else remaining the same. G becomes $G - dG$, and R, $R - dR$. Thus $G/(R + R')$ changes by $\{dG - GdR/(R + R')\}/(R + R' - dR)$. If we make the thickness of the layer very small, $G/(R + R')$ will be the same whatever layer is removed, and may in that case be regarded as a constant, and as we are considering only the effect of a particular layer we consider $R + R'$ as a constant. We have, then, to find the value of $a + c$ for which $dG - GdR/(R + R')$ is a maximum. If $a + c$ be denoted by u the necessary condition is

Bobbin with Graded Wire.

$$\frac{d}{du}dG - \frac{G}{R + R'}\frac{d}{du}dR = 0$$

or

$$\frac{\dfrac{d}{du}dR}{\dfrac{d}{du}dG} = \frac{R + R'}{G}$$

Performing the differentiations on the values of dG and dR, given in (18) and (19) above, we find

$$\frac{\rho p^2}{\pi a^2}\left(1 + \frac{u}{a}\frac{da}{du}\right) = \frac{R + R'}{G} = \text{constant} \quad . \quad . \quad . \quad (20)$$

If the radius of the wire and the thickness of its covering have always the same ratio, that is if u/a is constant, we have $a/u = da/du$, or $u/a \cdot da/du = 1$. Hence in this case a is in simple proportion to p.

On the other hand if the thickness of the covering is always the same, $da/du = 1$, and we have $p^2(2a + c)/a^3 =$ constant.

On the first supposition, denoting a by αp and $a + c$ by βa, where α and β are constants, and putting $-N/q$ for the integral of the term depending on the chamber in which the mirror hangs, we find from (18)

$$G = \frac{N}{4\alpha^2\beta^2}\left(\frac{1}{q} - \frac{1}{p}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where p is the greatest parameter used for the coil. In general q depends also on this value of p, but, as will be seen from the

figure, is nearly constant if the chamber is not large. It is a quantity of the order of magnitude of the internal dimensions of the chamber, and may be regarded as the parameter of the curve which would generate by revolution round the axis a volume equal to that of the needle chamber.

Outer Layers Comparatively Ineffective. Effect of Spires near the Axis of Coil.

We see from (21) that very little is gained, when this mode of winding with graded wire is adopted, by making p large in comparison with q.

If the chamber in which the needle hangs is cylindrical and runs right through the coil, the needle is shorter than the diameter of the smallest spires, and every spire in the coil produces an effect in the same direction on the needle. If however the space in which the needle hangs is not made cylindrical, the shape of it is of some importance, as it is

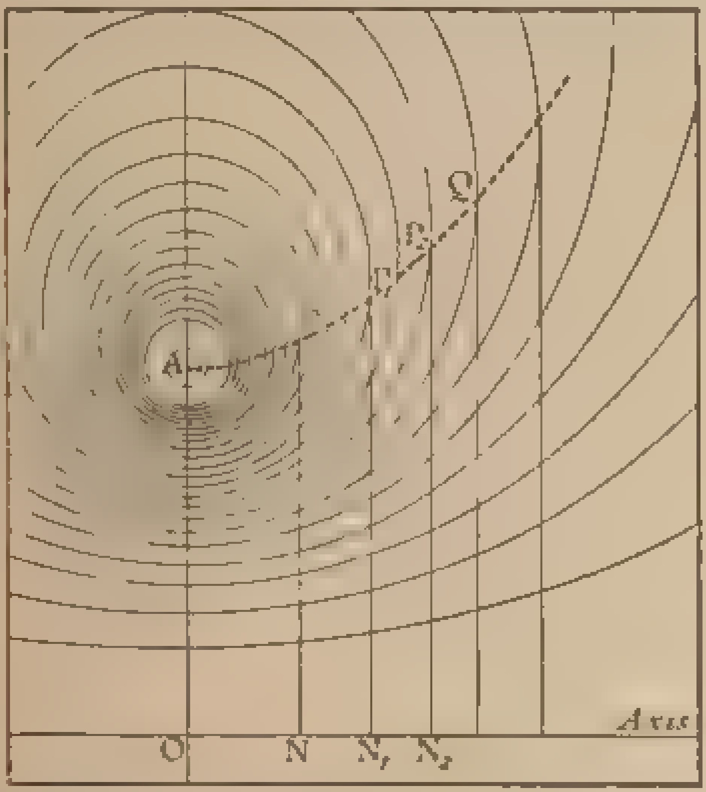


Fig. 79.

possible to place spires in positions in which they produce a magnetic effect opposed to that of the coil generally.* To see

* This is pointed out in Messrs. Ayrton, Mather, and Sumpner's paper, *Phil. Mag.* July, 1890.

this it is only necessary to consider the diagram of lines of force (Fig. 79) due to a single turn of wire of radius OA. Take any line of force and draw a tangent, PN, to it at right angles to the axis. Then it is clear that a uniformly magnetized needle at right angles to the axis, half of which is represented in position and length by PN, will not be acted on by any couple, since the force on each pole is in the direction of the length of the magnet. If however the magnet be at a greater axial distance, the force upon it is in the same direction as it would be if the needle were very short. Thus on a needle of the length and in the position here specified two turns, one smaller, the other larger in radius than the turn shown in the diagram, and in the same plane with the latter, would, if traversed by currents in the same direction, produce opposite couples. The smaller turn would however produce a couple in the same direction as the larger, if carried off to a sufficient axial distance from the needle.

Limiting Positions for Spires near the Axis.

For a needle of given length it is easy to draw a curve of limiting positions for the spires. For draw the line APQ through the points of contact of tangents perpendicular to the axis, then the axial distances ON_1, ON_2 of these tangents from the plane of the spire are the limiting distances of the spire from magnets of the half length N_1P_1, N_2P_2, &c. Then by supposing the scale of the diagram reduced in the ratio of N_2P_2 to N_1P_1 we shall have a spire of radius $OA \times N_1P_1/N_2P_2$, in the position to exert zero couple on a needle of half length NP when at an axial distance $ON_2 \times N_1P_1/N_2P_2$, and so for other points.

Form of Cavity for Needle.

It is therefore clearly undesirable to fill with spires wound in the same direction as the rest of the coil the space near the needles, beyond the limits indicated by these considerations. Figure 80* shows the form of the cavity which ought to be left. If it is possible to fill any of this space with wire, it should be done, but the spires made to run in the opposite direction, so that the couples due to their magnetic action may be in the same direction as that due to the rest of the coil.

Wiedemann's Aperiodic Galvanometer.

A form of galvanometer very convenient in many respects is that invented by Wiedemann.† A circular disk, or ring, of steel about 2 cms. in diameter, magnetized parallel to a diameter, is suspended with its magnetic axis horizontal and forms the needle of the instrument. This needle is attached to the lower end of a bar of aluminium, which also carries the mirror (made

* From Messrs. Ayrton, Mather, and Sumpner's paper, *Phil. Mag.* July 1890.

† *Die Lehre v. d. Elektricität*, vol. iii. p. 289.

of thin glass); and is hung within a damping chamber of copper, by a cocoon fibre, from a torsion head above, by means of which the effect of the torsion of the fibre can be estimated. The mirror is fixed so far above the needle that it is clear of the coils, and is viewed through a telescope in the ordinary

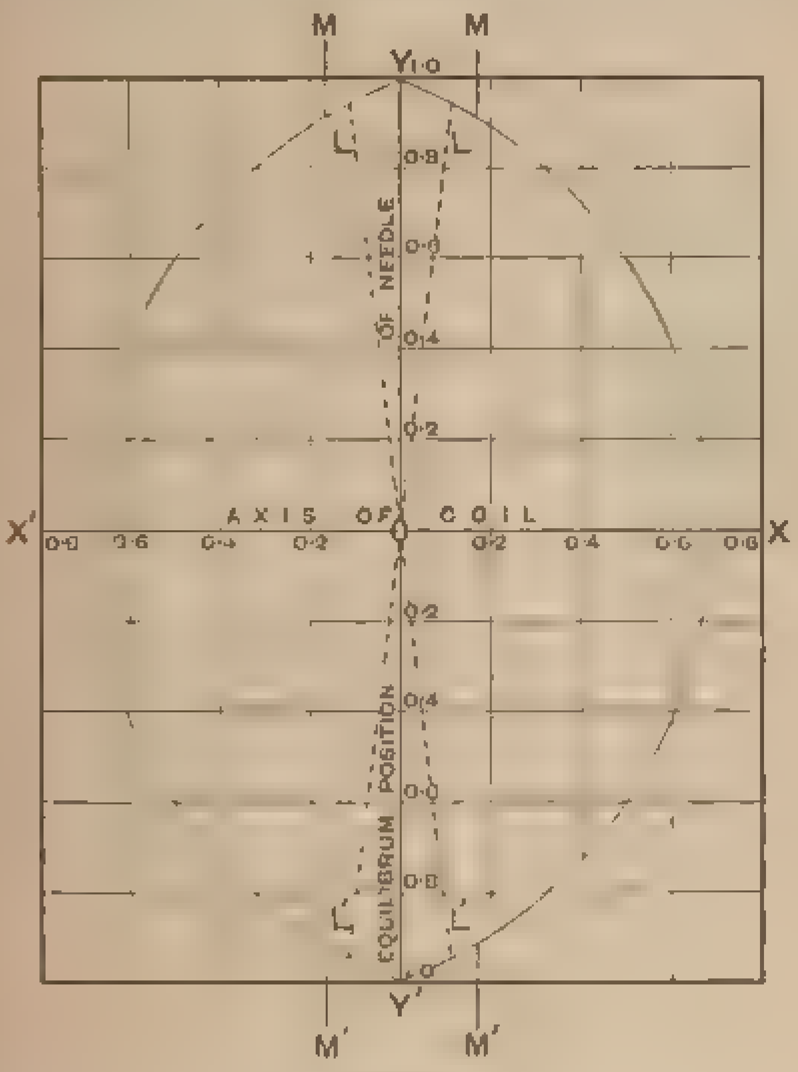


FIG. 80.

manner. The suspension fibre, aluminium bar, and attached mirror are protected by means of a glass tube and case fixed above the damping chamber.

A pair of coils is arranged, one on each side of the damping chamber, with their axes in line through the centre of the needle, and are attached to sliding pieces so that their

distances from the needle can be increased or diminished, and the sensibility altered accordingly. The openings in the coils are large enough to allow the bobbins to slide over the damping box close up to the needle, leaving, in the closest position, between them only the narrow space necessary for the tube down which passes the fibre

Two or three sets of pairs of coils suitable for different purposes are provided with the instrument. When the needle moves in the damping box of copper its motion is resisted by the action of the induced currents produced, so much so that it hardly oscillates about a new position of equilibrium

Action of Siphon-Recorder

In Sir William Thomson's siphon-recorder for registering signals sent through a submarine cable, a coil of wire is suspended between the poles of a magnet so as to be free to turn round a vertical axis passing through its centre Within the coil is fixed an iron core which serves to concentrate the field on the coil. When the coil is in the undeflected position the planes of its spires are parallel to the direction of the magnetic field; but when a current is sent through the coil it turns, in a direction depending on that of the current, so as to increase the magnetic induction through its circuits. A return couple is provided in the recorder by means of a bifilar suspension. The magnet is either a permanent horse-shoe magnet, or an electromagnet excited by a local current. The current from the sending station passes round the coil, which, turning in one direction or the other according as a "dot" or "dash" is being indicated, actuates the writing siphon.

The ordinary dead beat reflecting galvanometer invented by Thomson for cable signalling and ordinary testing is described at p. 308, Vol I.

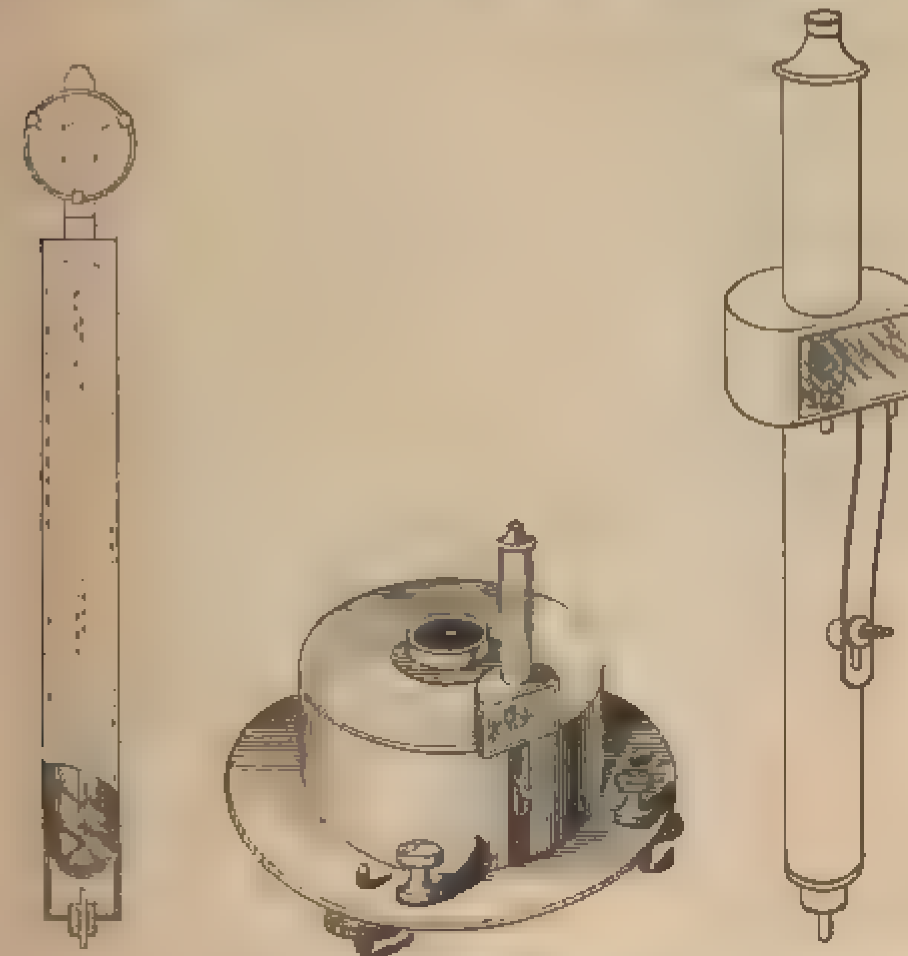
D'Arsonval Galvanometer.

The application of this arrangement as a galvanometer was pointed out in the first edition of Maxwell's *Electricity and Magnetism*, and has occurred to and been used by several experimenters. MM d'Arsonval and Deprez have however brought such instruments into general use for several purposes connected with practical electric work. The coil is hung by or rather strung on a stretched metallic wire, by which the current enters and leaves, and the torsion of this wire gives the required return couple. A core of iron is sometimes used within the coil as in the siphon-recorder. This if used at all should be quite independent of the coil, so that the coil may be adjusted relatively to the core, and pole-faces of the magnet. A mirror attached to the coil enables the deflections to be measured in the ordinary way.

Advantages.

This form of galvanometer possesses some advantages. It

can be made very sensitive by increasing the intensity of the field, and the coil possesses dead-beat quality in a high degree in consequence of the damping action of the induced currents produced in it when it is moving in the field. (See Chap. XIII.) It is moreover only to a slight extent directly affected by



a Coil Tube and Mirror, full size.

General view of instrument, showing circular permanent magnet with gap between poles for coil.

b Outer brass tube full size.

Fig. 81.

external magnetic bodies, since these unless very highly magnetized can only slightly affect the field in which the coil is placed.

An improved form due to Messrs. Ayrton and Mather is shown in Fig. 81. The coil is enclosed in a silver tube hung by a flattened wire of phosphor-bronze, with spiral of phosphor-bronze for lower connection.

It is desirable that the magnetic field of such a galvanometer should be as little disturbed as possible, in a manner at least which cannot be completely taken account of, and hence the use of iron cores in the suspended coils is inadvisable. Messrs. Ayrton, Mather, and Sumpner* have found it possible to make such a galvanometer give deflections proportional to deflections by dispensing with the iron core, and fitting iron pole-pieces to the stationary magnets, so shaped that the moving coil cut lines of force always at the same rate as the deflection varied.

Best Shape of Coils in D'Arsonval Galvanometers. It has been pointed out by Mr. T. Mather† that in instruments such as this in which suspended coils are used in magnetic fields, these coils should be long and narrow, and that the cross-section at right angles to the axis should be two equal circles touching on the axis. To prove this, it is to be observed first, that if the magnetic moment contributed by any portion of the wire be made greater by increasing the breadth of the spire in which it is placed, the moment of inertia of that part is increased in a greater ratio, and thus the period of free vibration of the coil is increased. The period of the coil is generally limited by practical requirements, and we have therefore to consider what the form of the coil should be, so that for a given moment of inertia there may be a maximum magnetic moment, or for a given magnetic moment a minimum moment of inertia. The solution is the same for both these cases. Consider (Fig. 82) an element E, of area dS, of a cross-

FIG. 82.

section in a plane at right angles to the axis A, and let n be the number of turns per unit of area. If γ be the current in each the current crossing ds is $\gamma n dS$. The couple round the axis exerted on unit of length of this part of the coil parallel to the axis is $\gamma n dS \cdot \mathbf{H}\, r \sin\theta$, where $\mathbf{H}$ is the intensity of the magnetic field, r the distance of the element from the axis, and θ the angle between AE and $\mathbf{H}$. If ρ be the average density of the coil, the moment of inertia of unit length parallel to the axis, and having

* *Phil. Mag.* July, 1890. † *Phil. Mag.* May, 1890.

the section dS, is $\rho r^2 dS$. The ratio of couple to moment of inertia for this part is thus $\gamma n \mathbf{H} \sin\theta/\rho r$, and this is to be made a maximum for every element of the coil. Thus $\sin\theta/r$ is to be made a maximum, since the other quantities are constant. The ends of the coil are ineffective as regards magnetic action, and hence so far as they are concerned it is desirable to make the distance of each element from the axis as small as possible. It is also desirable that the poles should be close in order to ensure with ordinary magnets as intense a magnetic field as possible.

Best Shape of Section of Coils Two Circles Touching at Axis

Consider now the curve the equation of which is

$$r = c \sin\theta \tag{22}$$

where c is a constant. A family of such curves can be drawn for different values of c, and they are all circles touching in the point A. Now let an element of wire be carried from the surface fulfilling this equation to a point lying outside. For such a point $\sin\theta/r$ has a smaller value. For a point lying inside $\sin\theta/r$ is greater. Thus, if the cross-section of the coil be filled up within any circle $r = c\sin\theta$, a diminution of the value of $\sin\theta/r$ would be produced by transferring any portion of the wire to any other unoccupied position.

The coil should therefore be made long in the direction of the axis, and have the form of cross-section shown in Figure 83,

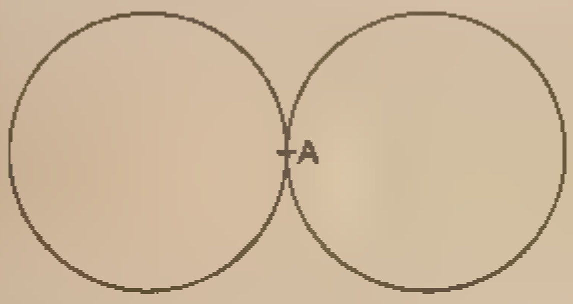


Fig. 83.

namely, two circles touching on the axis at the point A. The pole-faces should also be correspondingly long, and be broad enough to give a nearly uniform field at the coil, if they are not shaped so as to accomplish the object stated above.

Suspension of Coils.

The passage of the current along the suspension wire is apt to seriously affect the constant of the instrument, by altering its torsional rigidity. Suspensions made of twisted strips of thin phosphor bronze have been used by Professors Ayrton and Perry in several of their well-known instruments. These have

small torsional rigidity and great radiating surface, and are therefore peculiarly well adapted for use as torsion suspensions which at the same time act as conductors.

It has been pointed out in this connection by Messrs. Ayrton, Mather, and Sumpner that by making both coil and suspension of platinum-silver compensating effects are produced. If the rise of temperature were the same both in the coil and the suspension there would be exact compensation, since the percentage increase of resistance of platinum-silver is nearly equal to its percentage diminution of torsional rigidity.

The temperature variation of resistance is very slight in the case of the alloy called platinoid, now much in use for galvanometer and other coils, and on this account Mr. Mather* strongly recommends its use for the suspended coils of D'Arsonval voltmeters, and of rheostats for use with such coils.

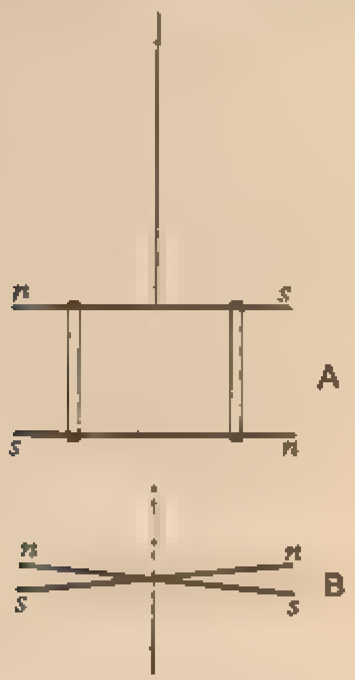


FIG. 84.

Astatic Galvanometers

In order to obtain sensibility, galvanometers are frequently made with astatic needles, that is suspended needle-systems which in a uniform field are either in equilibrium in any position or experience only a comparatively slight directive action. An astatic system generally consists of two similar horizontal needles

* *Electrician*, Jan. 8, 1892.

of equal magnetic moment arranged parallel to one another with their poles turned in opposite directions, as at *A*, Fig. 84, so that the resultant couple on the system is zero or very nearly so. Most commonly the needles are placed horizontally, as nearly as possible in the same vertical plane, with their centres in the same vertical line. In general however the needles are not quite parallel, and the system behaves like a needle of very small magnetic moment with its axis parallel to the line bisecting the obtuse angles between the projections of the needles on a horizontal plane as shown at *B* in Fig. 84. It has therefore been supposed that this is the manner in which an astatic system properly acts, but if this were so the sensibility of the arrangement would be entirely a matter of accident. When the system is so used moreover it is affected by the slightest external magnetic influence, and is a source of great trouble through the difficulty of maintaining a definite zero position. Astatic System of Needles.

An astatic system when quite accurately made has the needles exactly in one plane, and has almost perfect astaticism in a uniform field, and the sensibility is obtained by producing, by means of a magnet placed at some distance, a resultant magnetic field which is not uniform over the needle system, and therefore gives a differential action which furnishes the necessary directive force on the needles. An astatic galvanometer with directing magnet is shown in Fig. 85.* The instrument illustrated is one form of Sir William Thomson's astatic reflecting galvanometer. The details of the supports of the coils, needles, &c. will be clear from the Figure: the coils as will be seen are hinged so as to turn back to allow the suspended system to be easily got at. Each needle-system is a group of short needles, and there are two sets of coils, one containing each group of needles, and joined in such a way that the actions on the needles conspire. Sometimes a single coil only is used enclosing one of an astatic pair of needles. In this case although the coil exerts couples in the same direction on both needles, the principal turning action is exerted on that which is inside the coil. Perfectly Astatic System in Differential Field. Thomson's Astatic Galvanometer.

Another arrangement of astatic galvanometer is shown in Fig. 86. It is a slight modification of one adopted by Prof. T. Gray and the author for a very sensitive galvanometer constructed for the determination of the specific resistance of glass.† Gray's Astatic Galvanometer.

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* This cut has been kindly supplied by the Cambridge Instrument Making Co.

† *Proc. R.S.* No. 230, 1884. A similar arrangement of needles has, it appears, been used also by Herr Rosenthal and by Lord Rayleigh. See Ayrton, Mather, and Sumpner, *loc. cit.*

FIG. 86

The needles are a pair of horseshoes of hard steel as shown in Fig. 87, and are arranged in two parallel vertical planes so that the poles of one enter the cores of one pair of the four coils C, C the poles of the other the cores of the other pair of coils. The four coils are fixed in a plate with their axes parallel, and their faces in one plane; and the horseshoes are connected by a

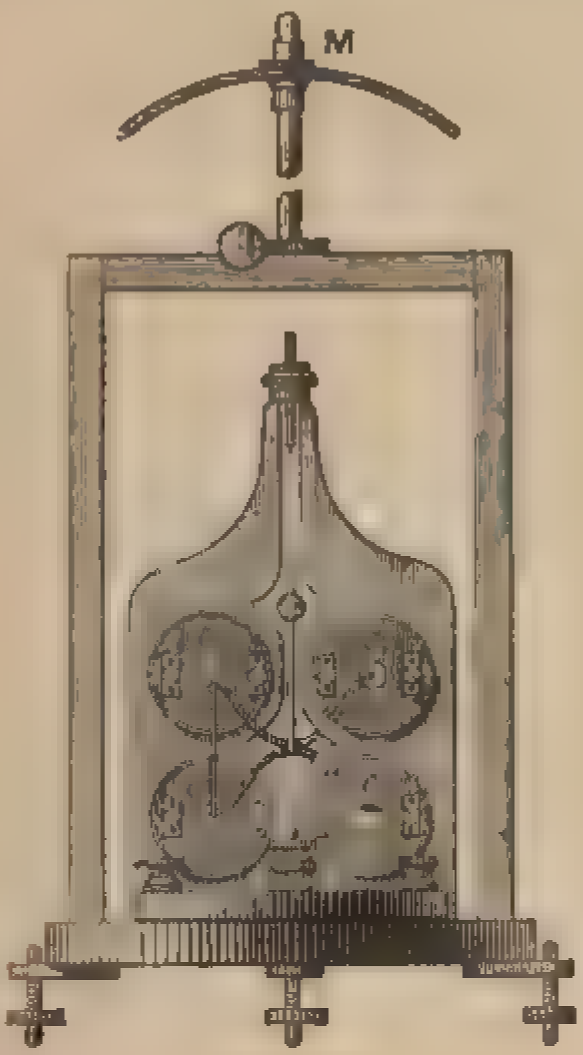


FIG. 86.

curved bar of aluminium so that one enters from one side of the coil system, the other from the other side as shown by the horizontal section in Fig. 87. The instrument is supported on a plate of vulcanite standing on vulcanite feet to give insulation, and the coils were wound on vulcanite bobbins. The coils are joined so that when a current passes both horseshoes are dragged

further into their coils, or both pushed out at the same time The needle-system is thus turned, and the deflection is measured by means of a mirror and scale in the usual manner The total resistance of the four coils was approximately 30,000 ohms; and the highest sensibility obtained when the instrument was set up was such that a current $1/10^{11}$ ampere produced a deflection of

FIG. 87.

1 division on a scale at about a metre distance The period of the coil was however for many purposes inconveniently long.

A very elaborate instrument on this principle was made for the Central Institution, London, from drawings made by Prof. Ayrton in consultation with Prof. T. Gray. A full description will be found in the paper of Messrs. Ayrton, Mather, and Sumpner above referred to.

The chief advantage of the arrangement of coils and needles described above is that a great portion of the wire of the coils is placed very near to the poles of the needles, and in a very favourable position for exerting the electromagnetic action required. The instrument, particularly the form shown in Fig. 2, is very easily made, and it does not cost more than an instrument of the ordinary kind. Of course a single horseshoe, or *S* or *Z* shaped bar, might be placed horizontally, and acted on by a pair of coils, and the principle thus applied to a single needle non-astatic instrument. In astatic instruments, however, of this form it is decidedly preferable, as shown below, to use vertical needles.

Vertical Astatic Needles. It is to be observed that if the line joining the poles or centres of gravity of magnetic polarity in each horseshoe be vertical, the system is perfectly astatic for a uniform field, for each vertical horseshoe is itself perfectly astatic. The pair of horseshoe needles can thus be adjusted to have as nearly as may be perfect astaticism in a uniform field, and thus made to preserve a nearly constant zero when under directive force, a result which it is exceedingly difficult to obtain in the ordinary arrangement of horizontal needles and which certainly rarely exists when a

horizontal magnet or magnets placed above or in an unsymmetrical position relatively to the needles is employed to regulate the sensibility, as then one of the needles must be magnetized and the other demagnetized to a greater or less extent, depending on the position of the magnet. According to this latter arrangement, if we suppose the needles to be parallel or nearly so, and $\mathbf{H}$ to be the magnetic field intensity at the upper needle, $\mathbf{H}'$ that at the lower needle in the same direction, m the magnetic moment of the upper needle, m' that of the lower needle, γ the current flowing, θ the deflection produced, and K a constant, we have—

$$\gamma = K \frac{\mathbf{H}m - \mathbf{H}'m'}{m + m'} \tan\theta \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Sensibility of Astatic Galvanometer.

The sensibility of an astatic instrument with horizontal needles as measured by the tangent of the deflection-angle for a given current is thus very great, as $\mathbf{H}m - \mathbf{H}'m'$ can be made, and is generally, very small. According to the values of m, m', $\mathbf{H}$, $\mathbf{H}'$, the instrument may or may not be seriously affected by external magnets, accidentally displaced in the neighbourhood of the instrument, or by slight changes otherwise caused in the magnetic field. It has been argued that since $\mathbf{H}$, $\mathbf{H}'$ (which are nearly equal) have each a considerable value, any slight magnetic disturbance producing only a very small percentage of change in each of these quantities cannot sensibly affect the value of the sensibility.

This however is a fallacy, as when the instrument is very sensitive, and $\mathbf{H}m - \mathbf{H}'m'$ is therefore very nearly zero, an exceedingly feeble magnetic disturbance changing $\mathbf{H}$ and $\mathbf{H}'$, as it will generally do, by the same absolute amount, and hence in very slightly different proportions, may suffice to alter $\mathbf{H}m - \mathbf{H}'m'$ by an amount comparable with its former value. The equilibrium position of the needles, for zero or any given current, will thus be subject to variation.

Slight changes in all or any of the quantities m, m', $\mathbf{H}$, $\mathbf{H}'$ may, therefore, affect the constant of the ordinary imperfectly astatic instrument very seriously, and as a matter of fact its constant has to be continually redetermined, for it is very sensitive to magnetic disturbances in the neighbourhood.

Advantages of Arrangement.

In the case, however, of needles adjusted to be accurately vertical these disadvantages do not exist. The needles retain their astaticism for uniform field and cannot be affected in the same way by directing magnets. Then $\mathbf{H}$, $\mathbf{H}'$ being the horizontal field intensities at the upper and lower extremities of the needles, γ the current strength, θ the deflection of the needles,

and K a constant depending on the coils, we have approximately—

$$\gamma = K(\mathbf{H} - \mathbf{H}') \sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The sensibility of the instrument can, therefore, be increased to any desired extent by placing the magnet M (Fig. 86) at a greater distance from the needles (or by counteracting its action by a smaller magnet placed nearer to the needles) so as to make $\mathbf{H} - \mathbf{H}$ sufficiently small. Further, variations of the strength of the horseshoe needles produce no effect unless they consist of changes of magnetic distribution, which may produce a deviation from perfect astaticism. When the instrument is properly adjusted and the needles are as nearly as possible uniformly magnetized, but little disturbance of this kind can be produced by the magnetizing action of the coils, since both poles of each have their magnetism augmented or diminished at the same time in the arrangement of Fig. 86, or both poles of one are magnetized more intensely in some degree, and both poles of the other weakened if both needles enter the coils from the same side.

Effect of Variations of strength of Needles.

Another possible arrangement of such a system of needles is with like poles turned in similar directions. The system will still be perfectly astatic if properly adjusted; and to give a return couple towards a zero position a magnet may be used, placed, for example, horizontally in the vertical plane at right angles to the front of the instrument, in a line passing through the suspension thread. If this magnet be placed nearer to say the lower ends than the upper ends of the needles, and the polarity of the end turned towards the needles be of the *same* name as that of the nearer ends of the needles, they will have a position of stable equilibrium when no current is flowing, with a horizontal line joining a pole of each needle at right angles to the direction of the magnet. The accurate law of variation of deflection with current is, however, in this case more complicated, and the instrument in some cases might have to be graduated by experiments with known currents of different amounts. Any change also of the magnetic distribution of the controlling magnet would affect the indications of the instrument.

Various arrangements of Vertical Astatic Needles.

It is to be observed that, in consequence of the horseshoe needles being placed in these instruments at a considerable distance from the axis of suspension, a very small value of $\mathbf{H} - \mathbf{H}'$ is sufficient to give the needle system such a directive force as to prevent any great error due to the rigidity or the viscosity of the suspending fibre.

The needle system may be hung in a uniform field, and a small needle rigidly connected with it, but placed so as not to

be perceptibly affected by the coils, used to give directive force to the magnetic system. This small needle may be hung in such a way that it can be turned round a horizontal axis at right angles to its length, and also round a vertical axis, so as to enable both the sensibility and the zero of the instrument to be adjusted. When the galvanometer is not intended for ballistic experiments, the frame on which the small needle is mounted may conveniently be immersed in a liquid and made to act as a vane for bringing the needle system quickly to rest. This arrangement, of course, would not be astatic, but would give great sensibility on account of the leverage of the horse-shoe needles as arranged.

Thus if m denote the magnetic moment of the small needle, H the horizontal component of the earth's magnetic force, k a constant depending on the coils, ϕ the strength of pole of each of the horseshoes (supposed of equal strength), and d the distance of these poles from the suspension thread, we have, since the deflection is small, for the turning couple exerted by the coils $4Ck\phi d$, and for the return couple $mH\theta$, and therefore—

$$C = \frac{mH\theta}{4k\phi d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Of course this arrangement is applicable whether like or unlike poles are turned in similar directions. It has the disadvantage that any change of m or ϕ or of both would affect the constant of the instrument.

The sensibility of any of these arrangements might also be increased by bringing out a very light arm, say from the middle of the cross-bar connecting the horseshoes, or from any other convenient point, and hanging the mirror by means of a bifilar, one thread of which is attached to the outer extremity of this arm, and the other to a near fixed point. The distance between the fibres being small in comparison with the length of the arm, small deflections would be greatly multiplied. This device would, no doubt, render a greater degree of skill and delicacy of manipulation necessary in the operator or experimenter, but it or some similar plan might in some cases be adopted, and the construction of these instruments renders its application to them very easy.

Astatic System with Straight Vertical Needles.

The astatic galvanometer described above may be modified as follows. Instead of a set of four coils with hollow cores and horse-shoe needles as described, eight coils may be used—one set of four arranged in rectangular order in a vertical plane facing a second set of four similar coils in a parallel plane at a small distance

from the first. Two *straight* needles of thin steel wire connected together as rigidly as possible by very light bars of aluminium, are so chosen as to length and so arranged that they hang from a single silk fibre with their lengths vertical and a magnetic pole as nearly as may be in the line joining the centres of each mutually opposite pair of coils. A magnet giving a differential field at the needles, if their like poles are turned in dissimilar directions, or any other arrangement may be used, and a current sent through the coils in any desired way by means of a distributing plate or otherwise.

Astatic galvanometers of Sir William Thomson's pattern are usually made with two coils, one above the other, split into four by a narrow vertical space in which the needle system is suspended, and which admits of the ready removal of the needles for adjustment. In this space may be hung, in a plane nearly parallel (when no current is flowing) to the two coils, two thin magnetic needles of steel wire side by side, kept with their lengths accurately vertical, and at a short distance apart (say $\frac{1}{4}$ or $\frac{3}{8}$ of an inch) by light aluminium, or other non-magnetic bars. Such a system of needles with unlike poles turned in similar directions would plainly experience a similar magnetic action to that exerted by the coils on the needles in the ordinary so-called astatic combination. But two straight vertical needles would plainly be perfectly astatic in a uniform magnetic field, and this astaticism for uniform field would not be liable to disturbance from any arrangement of magnets applied to give directive force to the system, as, for example, one or more magnets directing the system by means of a more powerful action at one end of the needle system than at the other, as shown in Figs. 85 and 86, or magnets arranged symmetrically with respect to both ends of the needles. An instrument with such a system of needles ought therefore to be subject to but slight, if any, disturbance in ordinary circumstances of sensibility when masses of steel or iron are being moved about at some little distance, and would we think be found useful in such cases, as for example in cable testing rooms.

A very sensitive galvanometer has been made for the Central Institution at South Kensington, under Prof. Ayrton's superintendence. Great attention has been given to details of arrangement, and specially good insulation has been obtained by supporting the coils on corrugated vulcanite pillars.*

Ballistic Galvanometers.

A ballistic galvanometer is an instrument designed for the purpose of measuring the whole quantity of electricity which

* See *Phil. Mag.* July, 1890

passes in a current of short duration. It is so called because the moment of inertia of the needle-system is made so great, and consequently the free period of vibration so long, that the current has begun and ended before the needle has sensibly moved from its initial position; just as in a ballistic pendulum the change of momentum of an impinging bullet has entirely taken place before the massive bob has moved from the position of stable equilibrium which it has under the action of gravity.

The arrangement of needles takes many different forms. For example Professors Ayrton and Perry constructed a ballistic galvanometer in which the needles were each a built-up sphere of small magnets*; the form of galvanometer referred to at p. 386 above was constructed for ballistic use, and several others on the same principle have been made for the same purpose; in other cases the needle is a disk of steel carefully polished to serve as mirror, and magnetized parallel to a diameter which is made horizontal when the needle is suspended.

The coil should always be set up so that the needles rest at right angles to its axis. This enables the needle if the deflection is kept small to be only slightly affected by the magnetizing action of the current in the coil.

The arrangements of coils is the same as in galvanometers for steady currents, except that on account of the influence of induced currents produced by the moving magnets the coils should be made with non-metallic cores or tubes; or if metallic tubes are used they should be slit longitudinally from end to end.

The siphon-recorder (or d'Arsonval Deprez) arrangement may also be used for ballistic purposes.

Theory of the Ballistic Galvanometer.

Let α be the initial angle which the needle makes with the plane of the coil, and θ_1 the angle which the needle would make with its initial position at the extremity of its deflection if there were no damping action. If M be the magnetic moment of the needle supposed short, and $G\gamma$ the magnetic force at the needle produced by a current γ in the coil, the turning couple on the needle is $M\,G\gamma\cos\alpha$. Hence if mk^2 be the moment of inertia of the needle, we have when the current is γ and the deflection from zero θ,

$$\frac{d^2\theta}{dt^2} = \frac{MG\gamma}{mk^2}\cos\alpha \quad . \quad . \quad . \quad . \quad . \quad (26)$$

* See Chap. XI. below.

If the whole current passes before there is any sensible deflection, we have, integrating over the whole time during which the current lasts,

$$\left(\frac{d\theta}{dt}\right)_{\theta=0} = \frac{MG\cos a}{mk^2}\int\gamma dt = \frac{MG\cos a}{mk^2}Q \quad . \quad . \quad . \quad (27)$$

if Q be the whole quantity of electricity which flows in the transient current.

Hence the kinetic energy given to the magnet is

$$\tfrac{1}{2}mk^2\left(\frac{d\theta}{dt}\right)^2_{\theta=0} = \tfrac{1}{2}\frac{M^2G^2\cos^2 a}{mk^2}Q^2 \quad . \quad . \quad . \quad . \quad (28)$$

1. Neglecting "Damping" of Needle.

This kinetic energy, as the magnet swings round and comes to rest in the magnetic field of horizontal intensity H, not necessarily that of the earth, is changed into magnetic energy of amount (see p. 6 above) $MH(1 - \cos\theta_1)$. Equating this to the value of the kinetic energy just found, we get

$$Q^2 = \frac{2mk^2\mathbf{H}(1 - \cos\theta_1)}{MG^2\cos^2 a}.$$

If T be the complete period of free vibration of the needle, we have $T = 2\pi\sqrt{mk^2/MH}$, or $mk^2/M = HT^2/4\pi^2$. Thus the last equation becomes

$$Q = \frac{HT\sin\frac{1}{2}\theta_1}{\pi G\cos a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

2. Taking into account "Damping" of Needle.

To take into account the damping action exerted on the needle by the air, &c., and by the induced currents produced in the coil by the motion of the needles, we shall suppose the deflection to be small enough to allow the sine of the deflection to be taken as equal to the angle, and take the retarding couple as proportional to the angular velocity, as it will be if the velocity is not too great. This theory will be sufficient, as the angular deflection can always be kept small, and nevertheless be read with accuracy; its smallness moreover prevents the angular velocity from becoming too great.

Let then the magnet make a small oscillation in the field of intensity H, and under the influence of the damping couple $\kappa\, d\theta/dt$. The equation of motion is

$$\frac{d^2\theta}{dt^2} + \frac{\kappa}{mk^2}\frac{d\theta}{dt} + \frac{MH}{mk^2}\theta = 0 \quad . \quad . \quad . \quad . \quad (30)$$

of which the solution, if T_1 be the observed period under the influence of the damping, is

$$\theta = A exp(-\kappa t/2mk^2) \sin \frac{2\pi}{T_1} t \quad . \quad . \quad . \quad . \quad (31)$$

where A is a constant, and t is reckoned from the instant of passing through the undisturbed position. T_1 is given by the equation

$$T_1 = \frac{2\pi}{\sqrt{\frac{MH}{mk^2} - \frac{\kappa^2}{4m^2k^4}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Equation (31) indicates simple harmonic motion of range diminishing in geometric progression as the time increases by successive intervals each equal to $T_1/2$. The Naperian logarithm of the ratio of any one amplitude to that which succeeds after an interval $T_1/2$ is $\kappa T_1/4mk^2$. This is called the logarithmic decrement of the motion and is generally denoted by λ. Thus Logarithmic Decrement.

$$\frac{\kappa}{2mk^2} = \frac{2\lambda}{T_1}.$$

From (31) we obtain

$$\frac{d\theta}{dt} = A . exp(-\kappa t/2mk^2)\left\{\frac{2\pi}{T_1}\cos\frac{2\pi}{T_1}t - \frac{\kappa}{2mk^2}\sin\frac{2\pi}{T_1}t\right\} . \quad (33)$$

If we write $2\pi/T_1 = R\cos e$, $\kappa/2mk^2 = R \sin e$, we have

$$\tan e = \frac{\kappa T_1}{4\pi mk^2} = \frac{\lambda}{\pi}, \quad R = \frac{2\pi}{T_1}\sec e \quad . \quad . \quad . \quad (34)$$

Using this value of $\tan e$, and the equation $4\pi^2/T^2 = MH/mk^2$, in (32), we obtain after reduction

$$T_1 = T \sec e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Equation (33) may now be written

$$\frac{d\theta}{dt} = \frac{2\pi A}{T_1}\sec e \,.\, exp(-\kappa t/2mk^2)\cos\left(\frac{2\pi}{T_1}t + e\right),$$

But when $t=0$, $d\theta/dt = MGQ/mk^2$, so that the last equation gives

$$A = \frac{MGQ}{mk^2}\frac{T_1}{2\pi}.$$

Thus

$$\frac{d\theta}{dt} = \frac{MGQ}{mk^2} \sec e \,.\, exp\,(-2\pi t \,.\, \tan e/T_1) \cos\left(\frac{2\pi}{T_1}t + e\right) \quad . \quad (36)$$

Putting in this $d\theta/dt = 0$, we get the value of t when the first deflection (or "throw") θ'_1 has just been completed. Thus $t = T_1\,(\pi/2 - e)/2\pi$. Hence (31) becomes for this value of t

$$\theta'_1 = \frac{T_1}{2\pi}\frac{MGQ}{mk^2}\, exp\left\{\left(-\frac{\pi}{2} + e\right)\tan e\right\}\cos e$$

$$= \frac{T_1}{2\pi}\frac{MGQ}{mk^2}\, exp\left(-\frac{\lambda}{\pi}\tan^{-1}\frac{\pi}{\lambda}\right)\cos e \quad . \quad . \qquad (37)$$

But if the oscillation were unretarded, and T the free period, we should have by (32)

$$\frac{MH}{mk^2} = \frac{4\pi^2}{T^2} = \frac{4\pi^2}{T_1^2}\sec^2 e = \frac{4}{T_1^2}(\pi^2 + \lambda^2),$$

or

$$mk^2 = \frac{MHT_1^2}{4(\pi^2 + \lambda^2)}.$$

Working Formula for Ballistic Galvanometer.

Substituting this value of mk^2 in (37), and solving for Q, we get finally

$$Q = \frac{HT_1}{2G}\frac{\theta'_1}{\sqrt{\pi^2 + \lambda^2}}\, exp\,.\left(\frac{\lambda}{\pi}\tan^{-1}\frac{\pi}{\lambda}\right) \quad . \quad . \quad . \quad (38)$$

This gives the first actual elongation θ'_1. If the damping be very slight so that λ is very small, we get approximately from (38) or directly from first principles, the equation

$$Q = \frac{HT_1}{2\pi G}(1 + \tfrac{1}{2}\lambda)\theta'_1 \quad . \quad . \quad . \qquad (38')$$

We shall have in Chapters X. and XI. below numerous examples of correction of observations of the effects of damping.

It is to be noticed that there is some uncertainty as to what the action of the air actually is when the needle of the ballistic galvanometer is suddenly set into motion. Any magnetizing or demagnetizing action on the needles must be as far as possible guarded against in the arrangement and use of the instrument. The deflection on this account ought to be always kept as small as possible, so that on the one hand the needle may never deviate far from the direction of the permanent field in which it is placed, and may on the other be always nearly at right angles to the axis of the coil, and thus only slightly exposed to magnetizing action in the direction of its length. Uncertainty of Ballistic Action.

The value of the ratio HG may be found by sending a steady current of known amount γ (determined by electrolysis as explained at p. 427 below, or by a standard galvanometer, or current balance) through the instrument and observing the deflection of the needle. If the indications follow the tangent law, and θ be the deflection, then $H/G = \gamma/\tan\theta$. Elimination of Constant, &c., for Ballistic Galvanometer

If the indications do not follow the tangent law the instrument can be calibrated by sending steady currents of different values through the coil, observing the deflections and interpolating for other currents by means of a curve plotted from the observations, or otherwise.

A condenser of known capacity C charged to a difference of potential V measured by some proper arrangement, may be discharged through the galvanometer and the deflection observed. This gives a known value of Q, and the value of HT_1/G can therefore be obtained by (38) or (38').

These methods and others will be exemplified below, especially in Chapters X., XI., and XII.

In cases in which the transient current can be repeated when desired, successive observations may be made without waiting for the needle to come to rest, by using the method of recoil proposed by Weber. The current is first sent in the positive direction round the coil, and the needle thereby caused to swing to its maximum deflection in the positive direction, then through zero to the negative side and back again to zero. At the instant when the needle arrives at zero the second time, the transient current is repeated but in the negative direction, thus reversing the motion of the needle, which swings to a maximum deflection on the negative side, then back again through zero to the positive side. When the needle returns to zero from the positive side, the transient current is repeated, but in the positive direction and so on, a fresh impulse being given in the opposite

Method of Recoil for Ballistic Observations.

direction to motion, every time the needle arrives at the zero position after a complete free swing from side to side. The angular deflections are shown in Fig. 88, which explains itself.

By equation (37) the first deflection θ_1 is given by the equation—

$$\theta_1 = \frac{2GQ}{HT_1}\sqrt{\pi^2 + \lambda^2}\, exp.\left(-\frac{\lambda}{\pi}\tan^{-1}\frac{\pi}{\lambda}\right) = KQ \quad . \quad (39)$$

When the magnet swings over to the other side, the numerical value of the deflection θ_2 will be given by

$$\theta \ = KQ\epsilon^{-\lambda}.$$

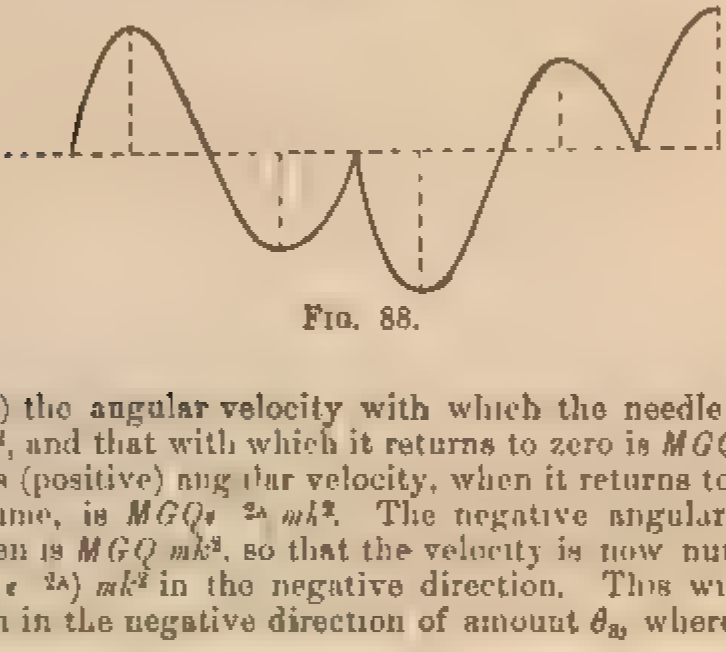


FIG. 88.

Theory of Method of Recoil.

By (36) the angular velocity with which the needle starts is $MGQ\ mk^2$, and that with which it returns to zero is $MGQ\epsilon^{-\lambda}/mk^2$. Hence its (positive) angular velocity, when it returns to zero the second time, is $MGQ\epsilon\ ^{2\lambda}\ mk^2$. The negative angular velocity given then is $MGQ\ mk^2$, so that the velocity is now numerically $MGQ(1-\epsilon\ ^{2\lambda})\ mk^2$ in the negative direction. This will give a deflection in the negative direction of amount θ_3, where

$$\theta_3 = KQ(1 - \epsilon^{-2\lambda}).$$

The next following amplitude will be positive, and will have the value

$$\theta_4 = KQ(1 - \epsilon^{-2\lambda})\epsilon^{-\lambda}.$$

Lastly, the velocity with which the needle returns to zero from the positive side, is $MGQ(1-\epsilon^{-2\lambda},\epsilon\ ^{2\lambda}\ mk^2$, and the positive velocity then imparted being $MGQ\ mk^2$, the velocity towards the

positive side is $MGQ\{1 - (1 - \epsilon^{-2\lambda})\epsilon^{-2\lambda}\}/mk^2$, and the deflection θ_5 is given by

$$\theta_5 = KQ(1 - \epsilon^{-2\lambda} + \epsilon^{-4\lambda})\,\epsilon^{-\lambda},$$

and so on.

We have for the first group of four deflections

$$\frac{\theta_4 - \theta_2}{\theta_3 - \theta_1} = \epsilon^{-\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (40)$$

and the same thing will be given by every succeeding group of four deflections. Hence, taking all such groups into account, we find

$$\frac{\Sigma\theta_4 - \Sigma\theta_2}{\Sigma\theta_3 - \Sigma\theta_1} = \epsilon^{-\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (41)$$

Combination of Results of Method of Recoil.

which gives the logarithmic decrement.

Again, from the values of the deflection found above, we have

$$KQ(1 + \epsilon^{-\lambda})(1 - \epsilon^{-2\lambda}) = \theta_3 + \theta_4,$$

$$KQ(1 + \epsilon^{-\lambda}) = \theta_1 + \theta_2.$$

Hence

$$KQ(1 + \epsilon^{-\lambda}) = (\theta_1 + \theta_2)\epsilon^{-2\lambda} + \theta_3 + \theta_4,$$

$$KQ(1 + \epsilon^{-\lambda}) = (\theta_3 + \theta_4)\epsilon^{-2\lambda} + \theta_5 + \theta_6,$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \qquad \cdot$$

$$KQ(1 + \epsilon^{-\lambda}) = (\theta_{4n-3} + \theta_{4n-2})\epsilon^{-2\lambda} + \theta_{4n-1} + \theta_{4n},$$

supposing $4n$ deflections to be observed. Adding the last set of equations, we obtain

$$KQ(1 + \epsilon^{-\lambda})$$
$$= \sum_{j=1}^{j=4n} \{(\theta_{j-3} + \theta_{j-2} + \theta_{j-1} + \theta_j)(1 + \epsilon^{-2\lambda}) - \theta_1 - \theta_2 - (\theta_{4n-1} + \theta_{4n})\epsilon^{-2\lambda}\} \quad . \quad . \quad . \qquad (42)$$

which enables Q to be found from a combination of all the observations made.

It is to be observed that this method cannot be conveniently used if the damping of the needle is very small, as then a regular repetition of successive sets of nearly the same amplitudes would be difficult to obtain. By observing the successive pairs of free elongations any change of zero which takes place during the experiments can be followed.

Formulae are easily obtained for taking into account the interval occupied in the passage of the current, if that is in the least comparable with the free period of the needle; but, as these are rarely necessary, we shall only give them if the need arises in connection with any electrical measurement described below.

Method of Successive Observation for Steady Current.

We only note further here that when a galvanometer is used for the measurement of a steady current, it may sometimes be desirable, in order to eliminate any variation of zero due to variation in the direction of the earth's force, to read the galvanometer as follows. The current sent round the coil of the galvanometer in the positive direction deflects the needle, which swings about the new position of equilibrium. The first, second, and third elongations are observed; then contact is broken for about half a whole period, so as to let the needle swing beyond zero; next the current is sent in the opposite direction to that in which it was sent at first, and the three first elongations on the other side observed; then the contact is broken, the current reversed, and so on as before.

If the numerical values of the first six deflections are $\theta_1, \theta_2, \ldots \theta_6$, we have for the deflection due to the steady current

$$\theta = \frac{\theta_1 + 2\theta_2 + \theta_3}{4} + \frac{\theta_4 + 2\theta_5 + \theta_6}{4}$$

or

$$8\theta = \theta_1 + 2\theta_2 + \theta_3 + \theta_4 + 2\theta_5 + \theta_6 \quad . \quad . \quad . \quad (43)$$

and so for any such series of six deflections.

Some account of methods of measuring currents, differences of potential, &c., in alternating circuits will be given in a later chapter. Many particular devices and arrangements which might have legitimately found a place in this chapter will be much more conveniently described in connection with the experiments in which they were originally used.

Lord Rayleigh's Current Weigher.

Lord Rayleigh and Mrs. Sidgwick have used in their researches on the electro-chemical equivalent of silver a form of electro-dynamometer balance, or current-weigher, in which the fixed and movable coils were placed with their axes coincident, and in such relative positions that the pull along the axis exerted by one coil-system on the other was a maximum. The fixed coils were the large coils of the British Association electrodynamometer described above, and between these was placed a coil of silk-covered wire wound on a ring of ebonite. The arrangement is shown in Fig. 89, which explains itself. We shall show that this coil placed midway between the two fixed coils was in

the position to have maximum force exerted upon it by each of the latter coils.

The use of a current weigher such as this has several important advantages over either the galvanometer or ordinary electro-dynamometer. As here arranged, the accuracy of the constant depended, in the main, only on the determination of the ratio of the radii of the coils; the necessity for finding H and taking account of its variations is avoided; and no difficulty as to the elastic or bifilar constant of suspensions exists. The actual observation of the indications is, however, a somewhat more elaborate process than in these other instruments, involving as it does an exact weighing. It can, however, be carried out with great exactness by a skilled experimenter.

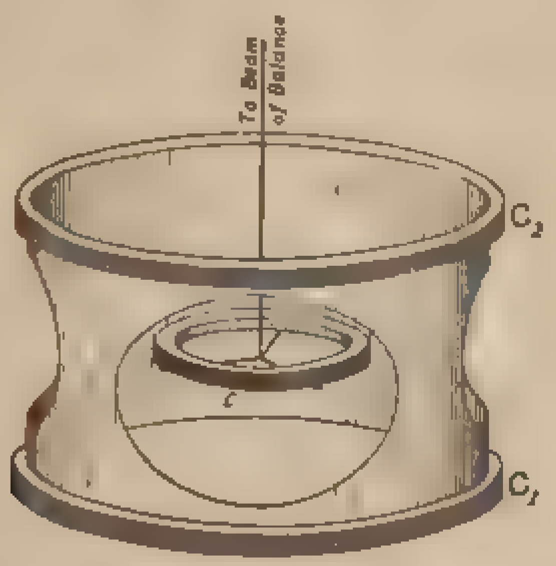


Fig. 89

The mutual electro-kinetic energy T_m of a system of two coils carrying a current γ is given by the equation

Attraction between Two Parallel Coils.

$$T_m = nn'\gamma^2 M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

where n, n' denote the numbers of turns in the two coils, and M denotes for the present the mean mutual inductance of a pair of turns one in each coil. Thus if x is the distance between the coils, the force F exerted by one on the other is given by

$$F = nn'\gamma^2 \frac{\partial M}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Force depends only on Ratios of Distances. It is well to notice here that $\partial M/\partial x$ is a mere number, and depends therefore only on the ratios a/α, x/a, (or x/α), of the radii of the coils, and of the radius of either to their distance apart. Thus if we write

$$\frac{\partial M}{\partial x} = f(a, \alpha, x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (46)$$

f is a homogeneous function of zero dimensions in a, α, x. Thus we have

$$df = \frac{\partial f}{\partial a} da + \frac{\partial f}{\partial \alpha} d\alpha + \frac{\partial f}{\partial x} dx \quad . \quad . \quad . \quad . \quad . \qquad (47)$$

with, by Euler's theorem of homogeneous functions, the condition

$$a\frac{\partial f}{\partial a} + \alpha\frac{\partial f}{\partial \alpha} + x\frac{\partial f}{\partial x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (48)$$

If the coils are so placed that the action between them is a maximum $\partial f/\partial x = 0$, and (48) gives

$$a\frac{\partial f}{\partial a} + \alpha\frac{\partial f}{\partial \alpha} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (49)$$

Thus by (47) equal (proportional) errors in the estimation of a and α produce no effect on the value of f provided the coils are in this position. Hence $\partial f/\partial x$ being zero there is (to quantities of the second order) no effect produced by errors in the estimation of x, and therefore the action between the coils depends only on the ratio a/α. This ratio, as will be explained below, can be determined electrically by a method due to Bosscha, without direct measurement of either a or α.

Expression for Force between Two Coaxial Circles. The value of M for different arrangements of coils is given in Chap. VI above. We shall use at present the expression given at p. 268 for the mutual induction of two coaxial circles of radii a, α, and distances x, ξ, from a fixed point on the axis. We have thus

$$\frac{\partial M}{\partial \xi} = \pi^2\frac{a^2\alpha^2}{r^4}\left\{1.2.3\frac{x}{r} + 2.3.4\frac{x^2 - \frac{1}{4}a^2}{r^3}\xi \right.$$

$$\left. + 3.4.5\frac{x^3 - \frac{3}{4}a^2x}{r^5}(\xi^2 - \tfrac{1}{4}\alpha^2) + \dots\right\} \quad . \qquad (50)$$

where $r^2 = a^2 + x^2$. Here a, ξ, are supposed to belong to the

small coil, and to be considerably less than a, x, respectively. Thus if a/α is not large, the value of $\partial M/\partial\xi$ will be given to a high degree of approximation by the first term alone of this series. Thus writing f' for $\partial M/\partial\xi$ we have, taking the first term only

$$f' = 1\,.\,2\,.\,3\,\pi^2 a^2 \alpha^2 \frac{x}{r^5},$$

and therefore

$$\frac{\partial f'}{\partial x} = 1\,.\,2\,.\,3\,\pi^2 a^2 \alpha^2 \frac{a^2 - 4x^2}{r^7}.$$

Thus $\partial f'/\partial x$ vanishes and the force is a maximum if $a^2 = 4x^2$, or $2x = a$, that is when the distance between the circles is half the radius of the larger.

Neglecting the second and third terms in (50) which involve ξ, and taking into account the part of the third term which involves α^2, differentiating and putting $x^2 = \frac{1}{4}a^2$ in all factors multiplying α^2, we get as a second approximation to the value of x for a maximum,

$$x = \tfrac{1}{2}a\left(1 - \frac{9}{10}\frac{\alpha^2}{a^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

Force on Movable Coil between Two Fixed Coils.

For two fixed coils at equal distances on opposite sides of the suspended coil the odd terms vanish, and we have (still supposing that the coils can be regarded as circles) for the action between one of the fixed coils and the movable one

$$\frac{\partial M}{\partial \xi} = \pi^2 \frac{a^2\alpha^2}{r^4}\left\{1.2.3\frac{x}{r} + 3\,.\,4\,.\,5\,\frac{x(x^2 - \frac{3}{4}a^2)}{r^5}(\xi^2 - \tfrac{1}{4}\alpha^2) + \ldots\right\} \quad . \quad (52)$$

The coils might then be arranged so that $x^2 = \frac{3}{4}a^2$, and thus to terms of the fourth order in α, ξ, the value of $\partial M/\partial\xi$ would be given by

$$\frac{\partial M}{\partial \xi} = 1\,.\,2\,.\,3\,\frac{2^4\sqrt{3}}{7^{\frac{5}{2}}}\pi^2\frac{\alpha^2}{a^2} = \cdot 2138 \times 6\pi^2\frac{\alpha^2}{a^2} \quad . \quad . \quad (53)$$

On the other hand, if, as was actually the case, $x = \frac{1}{2}a$,

$$\frac{\partial M}{\partial \xi} = 1\,.\,2\,.\,3\,\frac{2^4}{5^{\frac{5}{2}}}\pi^2\frac{\alpha^2}{a^2} = \cdot 2862 \times 6\pi^2\frac{\alpha^2}{a^2} \quad . \quad . \quad (54)$$

a considerably larger value. This equation multiplied by γ^2 gives a rough estimate of the force which would be produced by

a given current with two single turns, and therefore of the force to be expected between one of the fixed coils and the movable coil

By equation (50) when $\alpha = 2r$, we have, including two terms so as to find the effect of ξ^2 when this is not zero,

$$\frac{\partial M}{\partial \xi} = {\cdot}2862 \times 6\pi^2 \frac{\alpha^2}{a^3}\left(1 - 3\ 2\,\frac{\xi^2}{a^2}\right) \quad . \quad . \quad . \quad (55)$$

Effect of Error in Placing Movable Coil.

In the current-weigher used a was 25 cms., so that $\xi = 1$ mm. and neglected could only give rise to an error of about 1/20,000. Thus the instrument with ordinary care as to adjustment could be regarded as quite free from error due to inaccurate placing of the suspended coil.

Force on Movable Coil in Terms of Ratio of Coil-Constants.

As the ratio of the galvanometer constants was determined experimentally, and therefore was used in the calculations, we write down here the approximate expression for the force between one fixed coil and the suspended coil in terms of this ratio and the numbers of turns. Putting β for the value of the ratio we may write approximately

$$\beta = \frac{\frac{n}{a}}{\frac{n'}{\alpha}} = \frac{n}{n'}\,\frac{\alpha}{a}$$

or

$$\beta^2 \frac{n'^2}{n^2} = \frac{\alpha^2}{a^2}.$$

Relative Effects of Imperfect Insulation in Coils.

Thus approximately by (55) and (44)

$$F = {\cdot}2862 \times 6\pi^2 \frac{n'^3}{n}\,\beta^2\gamma^3.$$

An error in the estimation of n' the number of turns in the suspended coil, or, what is the same, any defect in the insulation, is thus of greater importance than a similar inaccuracy in the estimation of n.

The ratio β enabled the mean radius of the suspended coil to be calculated. The attraction between the coils was then found by an expression easily obtainable by differentiation from the value of M given in elliptic integrals at p. 142 above, for two coaxial circular conductors. Thus we have

$$\frac{\partial M}{\partial x} = \frac{\pi x \sin \xi}{\sqrt{\alpha a}}\{2F - (1 + \sec^2\xi)E\} \quad . \quad . \quad . \quad (56)$$

where $\sin\xi = k$. F and E have been calculated by Legendre, and were used by Lord Rayleigh in the formation of a table of values of $\pi x \sin\xi \{2F - (1 + \sec^2\xi)E\}$ for values of ξ proceeding by intervals of 6′ from 55° to 70°.

The value of $\partial M/\partial x$ was then found for the actual coils of axial breadths $2b$, 2β, $2d$, 2δ by employing the following formula of quadrature,* and multiplying by nn' the product of the number of turns. Thus $f(a, a, x)$ being the value of $\partial M/\partial x$ for a pair of mean turns, we have for the whole coils,

Calculation of $\frac{\partial M}{\partial x}$ by Formula of Quadratures.

$$\frac{\partial M}{\partial x} = \tfrac{1}{6}nn' \left\{ \begin{array}{c} f(a + d, a, x) + f(a - d, a, x) \\ + f(a, a + \delta, x) + f(a, a - \delta, x) \\ + f(a, a, x + b) + f(a, a, x - b) \\ + f(a, a, x + \beta) + f(a, a, x - \beta) \\ - 2f(a, a, x) \end{array} \right\} \quad . \quad . \quad . \quad (57)$$

We can now proceed to give an abstract of the experimental processes and results.

The suspended coil, C, of the current-weigher, which had been carefully wound with silk-covered wire on a ring of ebonite, was tested for insulation. The method adopted first was to make as nearly as possible an exact copy of the coil, then to place the coil and its copy side by side with their axes in coincidence, and join them in series so that a current could flow through them in opposite directions. A galvanometer with a needle of long period of free vibration was included in their circuit. One pole of a very long steel magnet was then thrust suddenly through the opening of the coils, and produced in them opposite induced currents, which, if the insulation had been perfect in both coils, ought to have together produced no effect on the needle of the galvanometer.

Mode of Using Current Weigher.

Tests of Insulation of Windings.

It was found however that the copy decidedly preponderated in magnetic effect; a result which pointed to faulty insulation in the ebonite coil. A comparison of the ratios of the self-inductions of the separate coils to the mutual induction of the pair in a fixed position confirmed this conclusion, and the coil was thereupon rewound.

After rewinding it was tested for insulation by a Hughes' induction balance. This consisted of two pairs of coils, one

Test by Hughes' Induction Balance.

* This formula it may be here remarked is applicable not only to M/x, but to any function of a, a, the mean radii. Thus it may be used to give M for two coils if $f(a, a, x)$ denote its value for the mean radii.

pair at some distance apart in one horizontal plane being joined up with a source of variable current in a primary circuit, the other pair in positions opposite the primary coils, and at distances finely adjustable by means of screws, being joined up with a telephone as a secondary circuit. When the coils had been adjusted to exact balance the introduction of a small circlet of copper ·004 inch in diameter between a primary and a secondary coil gave a very distinct sound.

The ebonite coil placed between one of the primary coils and its opposite secondary gave an audible sound, but much less than that occasioned by the copper circlet. When the ends were joined by a megohm of resistance the increase of sound was quite distinct; which showed that the insulation resistance was decidedly greater than a megohm, and therefore amply sufficient.

Particulars of Suspended Coil.

The particulars of the suspended coil were as follows:

Number of turns	242.
Radial depth 2δ	·9690 cm.
Axial breadth 2β	1·3843 cm.
Mean radius, found electrically as described below . . .	10·2473 cms.

The coil was made of copper wire insulated with silk saturated by paraffin wax. Its resistance was about $10\frac{1}{2}$ ohms.

Particulars of Fixed Coils.

The particulars of the fixed coils, C_1, C_2, as derived mainly from a record in Clerk Maxwell's handwriting in the Cavendish Laboratory note-book were as follows:

Number of turns in each . . .	225.
Mean radius, a	24·81016 cms.
Distance of mean planes, $2x$ . .	25·000 cms.
Radial depth, $2d$	1·29 cm.
Axial breadth, $2b$	1·50 cm.
Resistance of each coil (about) .	$14\frac{1}{2}$ B.A. units.

By measuring the distances from outside to outside, and from inside to inside, of the grooves filled with wire, the distance of mean planes was found to be 25 cms. exactly. The half-difference between these distances gave $2b = 1·5024$ cm. The mean radius and number of turns could not be verified, but the recorded value of the former agreed with the outside circumference, and the check on the counting of the number of turns given by the device adopted when the coil was being wound, of at the same time winding string on a drum turning with the coil, almost absolutely ensures the accuracy of the number given.

Experimental Determination of Ratio of Coil-Constants

The ratio of the radii was found as follows. One of the dynamometer coils, and the suspended coil, were made concentric and coaxial with their planes vertical in the magnetic meridian, and a small needle was hung at the common centre. A diagrammatic sketch of the arrangements is shown in Fig. 90. *D* is the dynamometer coil, *E* the ebonite coil, *N* a resistance box. When the thick copper piece *P* was made to join the mercury cups *F*, *H*, the current from a cell *A* was divided between the two coils, which were joined so that the current flowed round them in opposite directions. The reversing key *B* enabled the current to be sent first in one direction then in the other through the double arc.

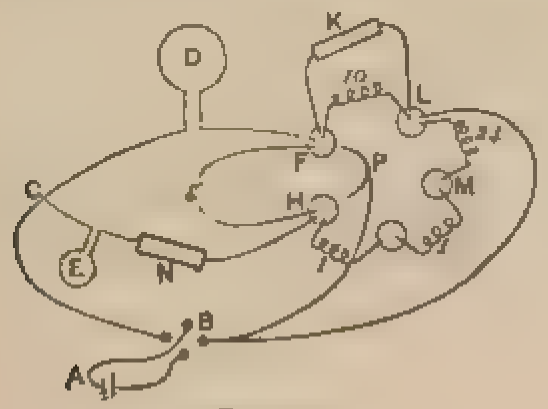


Fig. 90.

By means of *N* the resistances of the arcs joining *C* and *P* were adjusted so that no deflection of the needle took place. It was found that the resistance taken from *N* which gave balance could not be exactly determined, owing to inductive effects produced by the reversal of the current. Readings of the deflections of the needle were therefore taken for imperfect adjustments, with values of the resistances on opposite sides of the required value, and the value for balance obtained from these by interpolation.

Measurement of Balancing Resistances in Circuits of Coils.

The ratio of the resistances of the double arc was then obtained by making the two arcs adjoining branches of a Wheatstone bridge. This was done by withdrawing the copper piece *P*, which had the effect of converting the arrangement into a Wheatstone bridge of which one pair of adjoining branches were *D* and *E*, *N*, connected at *C*, the other pair a series of three resistance coils (composed of two single units and a 24 unit coil) and a coil of 20 units with its terminals connected by a high resistance coil *K*. These branches were connected with

one another at L, and with the other pair at the cups F, H. The battery terminals were attached at C, L, and those of a sensitive testing galvanometer, g, at F, H. Thus the ratio of the resistances was determined, and for one dynamometer coil was found to be on three different occasions 2·60087, 2·60098, 2·60113, or a mean of 2·60099. The same coil tested with another set of resistances gave on two occasions in like manner 2·60046, 2·60026, or a mean of 2·60036. The mean was thus 2·60067. For the other coil 2·60072 was found.

Calculation of Mean Ratio of Mean Radii. If G_1, G'_1 be the galvanometer constants of the two coils, γ, γ' the currents flowing in them when their conjoint magnetic effect at the centre was zero, we have $nG_1\gamma$ and $n'G'_1\gamma'$ for the magnetic effects due to the coils, and $nG_1/n'G'_1 = \gamma'/\gamma$. But if R, R', be the resistances of the branches, $\gamma/\gamma' = R'/R$, and therefore

$$\frac{nG_1}{n'G'_1} = \frac{R}{R'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

But using for each coil the value of G_1 given at p. 269 above, putting $x = 0$, $\xi = 0$, since it is the magnetic forces at the common centre that are in question, we find

$$G_1 = \frac{2\pi}{a}\left(1 + \frac{1}{3}\frac{d^2}{a^2} - \frac{1}{2}\frac{b^2}{a^2}\right)$$

$$G'_1 = \frac{2\pi}{\alpha}\left(1 + \frac{1}{3}\frac{\delta^2}{\alpha^2} - \frac{1}{2}\frac{\beta^2}{\alpha^2}\right)$$

or

$$\frac{\alpha}{a} = \frac{nR'}{n'R}\,\frac{1 + \frac{1}{3}\frac{d^2}{a^2} - \frac{1}{2}\frac{b^2}{a^2}}{1 + \frac{1}{3}\frac{\delta^2}{\alpha^2} - \frac{1}{2}\frac{\beta^2}{\alpha^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Now the known values of a, b, d, β, δ, and the approximately known value of α gave at once the value 1·001296 for the second fraction on the right of the last equation. Hence

$$a = \frac{225}{242} \times 2{\cdot}60070 \times 1{\cdot}001296 \times \alpha = 2{\cdot}42113\alpha \quad . \quad (60)$$

Adjustment of Suspended Coil. The suspended coil was adjusted in position in the current-weigher by first suspending it in a horizontal position, and then levelling and otherwise adjusting the positions of the dynamometer coils. A movable piece stood on three feet on the top of

the upper dynamometer ring, and in every position touched its inner cylindrical face in other two points. This piece was moved round the coil, and carried with it a pointer which thus described a circle coaxial with the fixed coils. When the latter were properly placed the pointer just played exactly round the outer surface of the suspended coil.

The level of the suspended coil was adjusted by carrying along the upper face of the upper dynamometer ring a straight rule provided with a pointer which just reached down and touched the upper surface of the suspended bobbin when that was in the proper position. The level of the dynamometer coils was changed until this point when moved about just scraped over the upper surface of the suspended coil.

Final Value of $\frac{\partial M}{\partial x}$.

The value of $f(a, a, x)$ was $\pi \times 1{\cdot}044576$. From this, by the table of values of the elliptic integral expression referred to above, the terms of the expression on the right of (57) were calculated and gave

$$\frac{\partial M}{\partial x} = \pi n n' \times 1{\cdot}044627 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

If in any experiment the current was γ, the attraction or repulsion between each fixed coil and the suspended coil was $nn'\gamma^2 f$. If m denote the observed difference of weights applied before and after the reversal of the current

$$4nn'\gamma^2 f = mg \times {\cdot}99986,$$

where ·99986 is the correcting factor for the air displaced by the weights m, and g is the acceleration produced by gravity at the place of experiment. This was taken as 981·2822 in centimetre-second units. Hence m being taken in grammes

$$\gamma^2 = \frac{981{\cdot}2282 \times {\cdot}99986}{4 \times 225 \times 242 \times 1{\cdot}044627} \frac{m}{\pi}$$

or

$$\gamma = {\cdot}037048 \sqrt{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

Sir W. Thomson's Standard Current Balances.

Sir William Thomson has constructed current-weighers or balances for use as standards for current measurement in practice, and as instruments on the principle of the balance have been adopted for the same purpose by the Board of Trade Committee on Electrical Standards (see their Report in Appendix) we give here a short account of the most generally useful form of these balances.

These instruments are based on the principle, set forth in Chap. III. above, of the mutual action between the fixed and movable portions of a circuit carrying a current. Each of the mutually influencing portions consists in most of the instruments of one or more complete turns or spires of the conductor, but in some cases consists of only half or part of a turn. In all cases in what follows we shall, following Sir W. Thomson, call each portion a *ring*.

In each of the balances, except that for very strong currents (the kilo-ampere balance), the movable portion of the conductor consists of two rings, carried with their planes horizontal at the extremities of a balance beam free to turn in the ordinary way round a horizontal axis. Above and below each ring on the beam is a fixed ring with its plane parallel to that of the movable ring. The rings are (except in what is called the Composite Balance *) all joined in series, and the current to be measured is sent through them so that the mutual action between the movable ring at one end and each of the two fixed rings there is to raise that movable ring, while the mutual action of the other group of three rings is to depress the corresponding movable ring. The action is therefore to turn the beam round the horizontal axis on which it is pivoted, with for any given position a couple varying as the square of the current flowing.

Arrangement and Action of Balance.

Fig. 91 shows diagrammatically the rings and the course of the current through them. a, e, b, f are the two pairs of fixed rings, c, d the movable rings. The current entering by the terminal T passes round all the rings in series, in the two movable rings in opposite directions, and returns to the terminal T_1. Since each movable ring is in general in a magnetic field, terrestrial or artificial, which has a horizontal component, it tends to set itself so that the greatest number of horizontal lines of force may pass through it (Chap. III. above) and therefore is acted on by a couple which tends to turn the beam round its axis. But since the current passes round the movable coils in opposite directions, and these are very approximately equal, the two couples are nearly equal and opposite, and the instrument is practically free from disturbance by horizontal magnetic force.

The turning couple produced by the mutual action of the fixed and movable rings is balanced for the horizontal or "sighted position" of the beam by an equal and opposite couple produced (in the manner more particularly described below), by a stationary weight at the end of the beam, and a sliding weight placed,

* See the Author's smaller Treatise, 2nd ed. p. 107.

steelyard fashion, at a suitable point on a graduated bar attached to the beam. The amount of the current flowing in the rings is deduced from the amount of the equilibrating couple thus applied, or rather from a number proportional to it, by means of a table of reckoning.

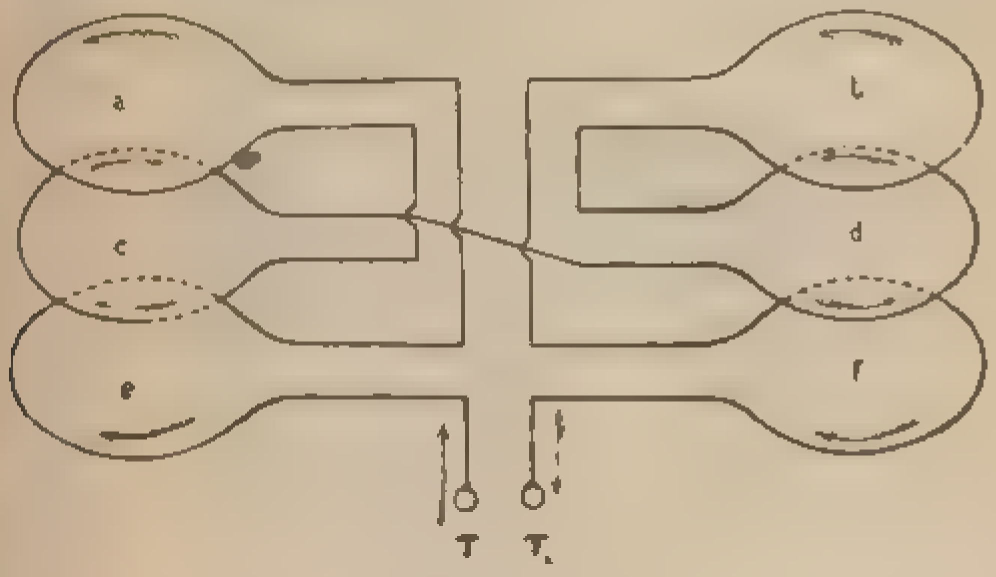


FIG. 91.

Most of the constructive details will be made out from Fig. 92 which shows the Standard Centi-ampere Balance, and illustrates the arrangement of the beam, the graduation, and the mode of applying the equilibrating couple, for all the instruments. Standard Centi-Ampere Balance.

The beam is hung on two trunnions, each supported by a flat elastic ligament made of fine copper wires, through which the current passes to and from the movable rings.

The horizontal or sighted position of the beam is that in which the pointers on the extreme right and left are at the middle divisions of their scales. This position, in all the instruments in which a movable ring is acted on by two fixed rings between which it is placed, is not that midway between these two rings, as that would be a position of minimum force and therefore of instability. For stability it is so chosen that the movable ring is nearer to the repelling fixed ring than to the attracting ring by such an amount as to give about ½ per cent. more than the minimum force.

Fixed to the beam and parallel to it is a finely graduated bar, and above this is a horizontal fixed scale, called the Inspectional Scale, less finely divided. Both graduations begin

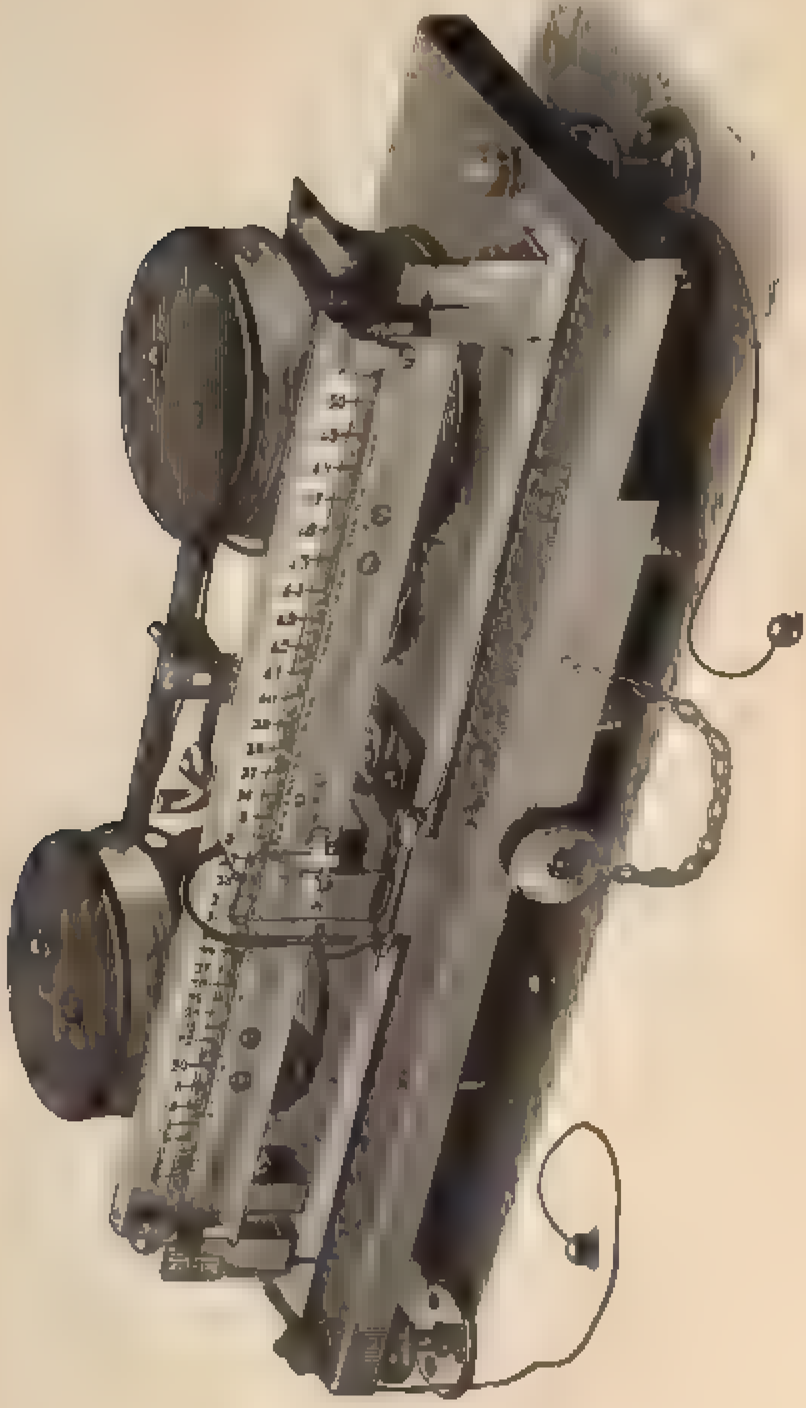


FIG. 92.—STANDARD CENTI-AMPERE BALANCE

from zero on the extreme left and have numbers increasing towards the right. A carriage is moved along the graduated bar to any required position by a sliding piece controlled by a cord which can be pulled from either end, and this carriage, by itself or with an additional weight, forms the movable weight referred to above. The position of the carriage is indicated by a pointer which moves along the lower scale. Each additional weight has in it a small hole and slot which pass over conical pins in the carriage. This ensures that the weight is always placed in a definite position. The balancing weight is moved along the beam by means of a self-releasing pendant carried by the sliding piece above referred to. To this pendant is attached a vertical arm (seen in the figure) which passes up through the recess in the front of the weight and carriage and so enables the carriage to be moved with the sliding piece. The stationary weight is placed in the trough shown at the right-hand end of the instrument. The trough is V shaped, and the weight cylindrical, with a cross pin which passes through a hole in the bottom of the trough. The weight is thus placed in a perfectly definite position and always has the same leverage. It is so chosen as just to keep the beam in the sighted position when the sliding weight is at the zero of the scale. Arrangement for Applying and Removing Weights.

Since the mutual action of the rings is to bring the beam towards the sighted position when displaced by the weights, and the equilibrating couple is that due to the displacement of the sliding weight from zero, the latter couple increases as the current increases, and hence motion of the sliding weight towards the right corresponds to increasing currents. The use of the stationary weight gives a scale of double the length which would be obtained without it. Device for Obtaining Long Scale.

In the top of the lower or finely graduated scale are notches which correspond to the exact integral divisions in the upper fixed scale. Thus the reading in the fixed scale is got when the pointer is at a notch, without error from parallax due to the position of the eye. The reading when the pointer is between two notches is easily obtained by inspection and estimation with sufficient accuracy for most practical purposes. When however the greatest accuracy is required, the reading is taken on the lower scale, with the aid of a lens, and the current strength calculated from a table of doubled square roots. Graduated Scale.

Four pairs of weights are given with each instrument. Of these one set is for the sliding platform, the other set are the corresponding counterpoises. The weights of each set are in the ratios 1 : 4 : 16 : 64, and are so adjusted that, when the Weights.

carriage is placed with its index at a division of the inspectional scale, the instrument shows a current of an integral number of amperes, half-amperes, or quarter-amperes, or some decimal subdivision or multiple of one of these units of current.

Adjustment of Zero.

The accurate adjustment of the zero is effected by a small metal flag as in a chemical balance. This flag is set in any required position by means of a fork moved by a handle beneath and outside the case of the instrument. The sliding weight is brought to zero with the corresponding counterpoise in the trough, and then the flag is turned to one side or the other until the pointer of the beam (seen on the extreme right and left in Fig. 89) is just at zero.

When necessary for transit or otherwise, the beam in the centi-ampere and deci-ampere balances is lifted off its supporting ligament by turning an eccentric by a shaft under the sole-plate of the instrument. In the other balances the beam is fixed for carriage by placing distance pieces between the upper and lower parts of the trunnions and screwing them together by milled headed screws kept always in position for the purpose.

Section II.

MEASUREMENT OF CURRENTS AND GRADUATION OF INSTRUMENTS BY ELECTROLYSIS.

Determination of Electro-Chemical Equivalent of Silver.

We shall now shortly describe the process of determining the electro-chemical equivalent of silver pursued by Lord Rayleigh and Mrs. Sidgwick. The arrangement of apparatus is shown in Fig. 92. A circuit was made up of a battery A in series with three silver voltameters, a tangent galvanometer D (which gave a rough measurement of the current), the current weigher F,G, described above (p. 398).* The voltameters were each composed of a platinum bowl which served as kathode, and an anode of pure silver plate suspended horizontally above

* The rest of the arrangements shown in Fig. 89 have no relation to the electro-chemical determination. They were required for the experiments on Clark cells described below (p. 433).

the bowl in the electrolytic liquid, which was a solution of pure nitrate of silver. To prevent disintegrated silver from falling from the anode the plate was wrapped round with pure filter paper secured at the back with sealing wax. The electrolyte was in general a neutral solution of 15 parts by weight of pure silver nitrate in 100 parts of water. The area of deposit in two of the basins was about 37 square centimetres, and 75 square centimetres in the other. Form of Voltameters.

After a number of trials of the addition of acetate of silver in small quantity to the pure nitrate solution, it was found that, while the acetate had the desired effect of giving a firmly coherent deposit of close texture, the very closeness of its texture rendered very difficult the after freeing of the deposit from retained salt or other impurity tending to increase its weight. It was therefore decided to use pure nitrate solutions, which it was found after all gave deposits coherent enough for the subsequent treatment.

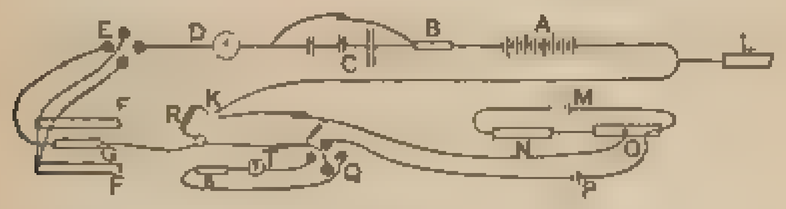


FIG. 93.

The procedure in an experiment was as follows. The current roughly regulated to the desired value was allowed to pass through the current weighing apparatus for half an hour, but not through the voltameters. The copper conductors of the circuit heated somewhat, and thus the current slightly fell off during this time. The voltameters in the meantime were charged with the solution, and the anodes fixed in position. Then when all had been adjusted the current was, at an instant observed on a chronometer, sent through the voltameters arranged in series, and the weights then required to bring the pointer of the suspended coil to zero were observed. At intervals the current was reversed, and the change of weights observed. For one direction of the current, of course, the electromagnetic action assisted gravity, in the other opposed it. Details of an Experiment.

The following table gives the result of a series of experiments made on March 10, 1884. The two sets of numbers are the weights which had to be added to give equilibrium according as the current was in one direction or the other. Result of Series of Experiments

Force on Current Weigher.

Time of Weighing. H.	M.	S.	Weight in Grammes.
4	19	30	7·694
4	25	0	6·795
4	32	15	7·698
4	40	20	6·791
4	42	50	7·699
4	50	30	6·790
4	53	10	7·699
4	56	30	6·789
5	1	15	6·789

Current sent through voltameters at 4h. 17m. interrupted at 5h. 2m.

Difference of weights = 2 × Force on suspended coil.

The curves, Fig. 94, show these results for each position of the key.

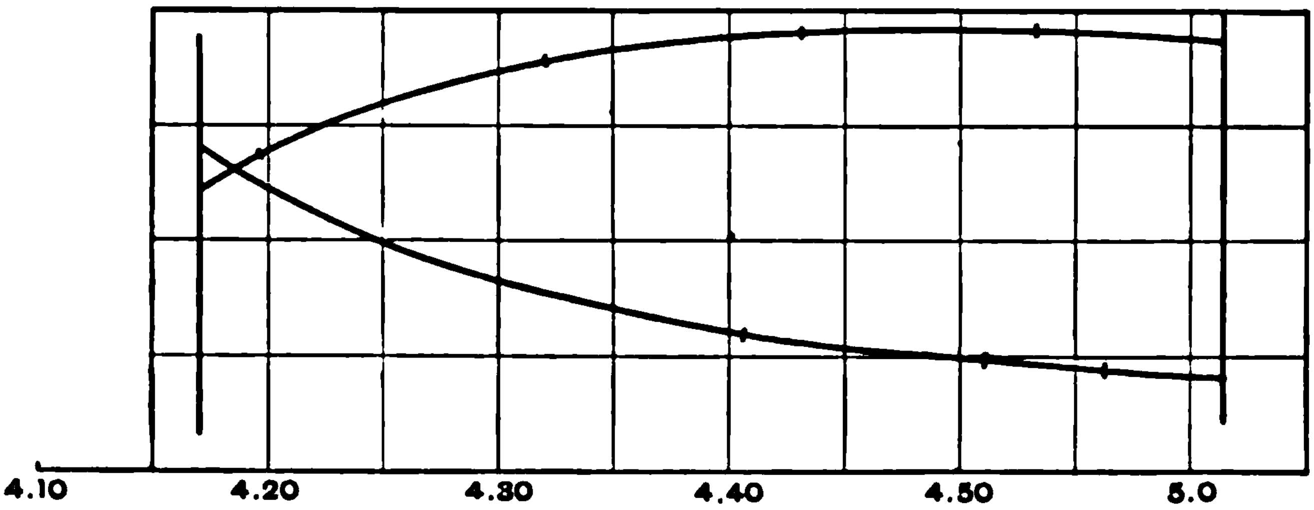


Fig. 94.

The current was integrated by dividing the whole interval of 45 minutes during which the current was flowing into 9 intervals of 5 minutes each, and the magnitude of the current at the middle of each interval was taken to represent its value during the period.

The differences of the ordinates of the curves of Fig. 94, at the middles of these intervals, give the difference of weights,

and therefore twice the force exerted by the fixed coils on the suspended one. These differences and their square roots are shown in the following table. The mean of the square roots is the square root of the difference of weights which would have been shown by the mean current.

Time. H. M. S.	Difference of Weights.	Sq. Root of Difference of Weights.
4 19 30	·897	·9471
4 24 30	·900	·9487
4 29 30	·904	·9508
4 34 30	·906	·9518
4 39 30	·908	·9529
4 44 30	·908	·9529
4 49 30	·909	·9534
4 54 30	·910	·9539
4 59 30	·910	·9539
		Mean ·95171

Correction for Time Lost in Reversals.

The 45 minutes' interval during which the experiment lasted was corrected for the time taken to work the reversing key. This was done by carrying the main current, between the battery and the key, round a reflecting galvanometer consisting of a few turns of wire. The momentary stoppage of the current caused the needle to fall back towards zero, and from the observed amount of the corresponding motion of the spot of light, and the period of the needle, the time of duration of the interruption could obviously be found. The correction rendered necessary was ·083 second for each operation. This brought down the whole interval by ·6 second, or to 2699·4 seconds.

Washing of Deposits

The deposits were washed immediately after formation first with alcohol, then with boiling water, and lastly with cold water. They were then left to soak in water overnight, then rinsed and put to dry in an air-bath at 160° C. After cooling over a desiccator the deposits were weighed, then were heated nearly to redness over a spirit lamp to drive off traces of adhering salt, then cooled and weighed again.

Results of Series of Experiments: Weights of Deposits.

The following table gives the results of the weighings for the set of experiments already referred to:—

March 10, 1884.

	Large bowl. I. Pure Nitrate Normal Strength.	Large bowl II. Pure Nitrate. Double Strength.	Small bowl. III. Pure Nitrate. Normal Strength.
Before deposit	80·4490 grms.	17·2958 grms.	21·8789 grms.
After deposit, first weighing	81·5138 "	18·3628 "	22·9434 "
Gain	1·0648 "	1·0643 "	1·0645 "
After strong heating . .	81·5135 "	18·3627 "	22·9433 "
Gain	1·0645 "	1·0642 "	1·0644 "

Mean gain 1·0644 grammes.

Thus the amount of silver deposited per second is 1·0644/2699·4. Dividing the mean square root of the difference of weights by this we get $\sqrt{m}$/(rate of deposition) = ·95171 × 2699·4/1·0464 = 2413·7.

Final Result of Several Series of Experiments.

The mean result of several series of experiments was to give instead of the last found number 2414·45. From this the value of the electro-chemical equivalent of silver was deduced. We have seen that if m is the difference of weights, we have

$$\gamma = \cdot 0370484\sqrt{m}$$

Electro-Chemical Equivalent of Silver

If w be the electro-chemical equivalent of silver, we have for the rate of deposit $w\gamma$. But

$$\frac{\sqrt{m}}{w\gamma} = 2414 \cdot 45.$$

Hence as final result

$$w = \frac{1}{2414 \cdot 45}\frac{\sqrt{m}}{\gamma} = \frac{1}{2414 \cdot 45 \times \cdot 0370484} = \cdot 0111794 \quad . \quad (63)$$

Dependence of Result on Strength of Solution and Current Density.

It is stated in the paper that the strength of the nitrate solution may be considerably varied without affecting the result if the current does not exceed $\frac{1}{4}$ ampere for the 37 sq. cms. area of deposit. In this case a 4 per cent solution may be used. If the currents are comparatively strong, the solutions should be from 15 to 30 per cent. in strength. Too weak a solution would give a somewhat loose deposit. Currents not exceeding $1\frac{1}{2}$ ampere can be conveniently measured by running them for about a quarter of an hour through a strong solution.

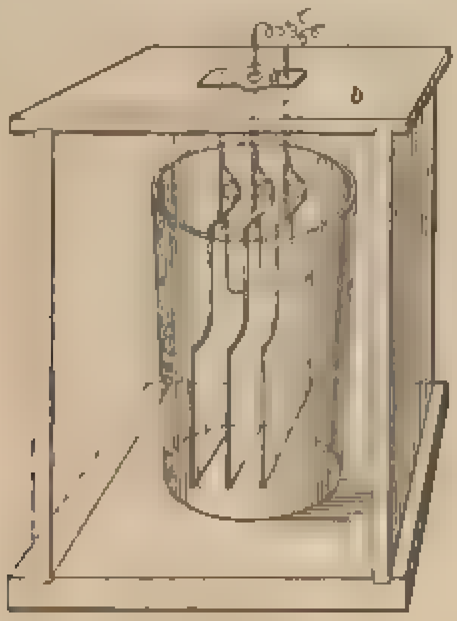


FIG. 95.

Measurement of Currents by Electrolysis of Copper Sulphate.

The graduation of instruments for use as standards in practical electricity is now carried out with great accuracy in Sir William Thomson's laboratory by means of the electrolysis of copper sulphate. The behaviour of this substance as an electrolyte, and hence the conditions necessary for obtaining consistent results in its use, and the ratio of the electro-chemical equivalent of copper to that of silver, were carefully investigated by Prof. T. Gray,* who was for some time in charge of the graduation of Sir W. Thomson's standard instruments, and a short account of his results is here given.

* See a paper on the "Electrolysis of Silver and Copper," T. Gray, *Phil. Mag.* Oct. 1886, from which the details here given are mostly taken. See also a paper by A. W. Meikle, *Electrical Engineer*, Mar. 23, 1888.

Form of Voltameter

A form of cell very convenient for use with solutions, whether of nitrate of silver or sulphate of copper, when the current strength is not greater than 10 amperes, is shown in Fig 95. It consists of three parallel plates of pure silver, or pure copper, suspended from spring clips in a glass vessel containing the proper solution. This form of cell has the advantages of giving light plates, which facilitate the accurate weighing of the amount of loss or gain of metal, and allowing, when silver is used, and the size of the plates is properly proportioned, the loss from the anode to be used as a check in estimating the gain on the kathode. There is of course the objection which attends the use of vertical plates,

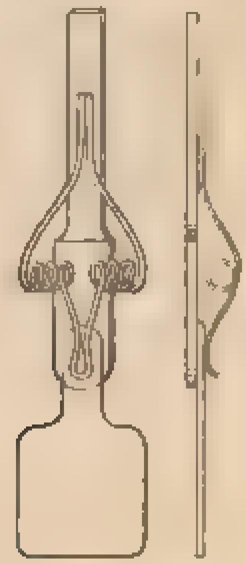


Fig. 96.

that the solution becomes less dense near the kathode, but the only practical effect due to this has been found to be a slightly greater thickness of deposit in the lower part of the plate due to the greater density there.

Lord Rayleigh has used, as explained above, as voltameter a platinum bowl as kathode, and a silver plate as anode. This cell, though it has several advantages, has been found, according to Prof. T. Gray's experience, more difficult to manipulate than that here described.

Form of Plate holder.

The form of clip or plateholder, as illustrated in Fig. 96, almost explains itself. It is made of stiff platinoid or brass wire. A piece is taken of the proper length, bent into a close loop at the middle, then each half wound two or three times

round a rod of metal to form springs as shown, and the two ends bent round to meet side by side, and there soldered to a stiff back-piece of brass. The springs when soldered in position should cause the loop to press firmly against the back-piece so as to form a firm clip.

The stems of the two outer clips when in position are connected by a cross-piece *a* of copper. Both are insulated from the inner clip by a block of vulcanite through which its stem passes. This whole arrangement of cross-piece and insulating block is fixed on the top *b* of the wooden framing shown in Fig. 95.

The two plates attached to the outer clips form the anode of the electrolytic cell, and the plate between them the kathode. The kathode thus gains on both sides, and as it is safer to use the gain than the loss of metal in estimating the current, the weight of the plate itself is thus made as small as possible in comparison with the alteration in weight to be determined.

Since the form of cell shown in Fig. 95 was arrived at, it has been improved by the substitution for the cover *b* of a rectangle of wood, well soaked in paraffin or varnished, which carries on one side the clips for the anode, and at the middle of the opposite side the single clip for the kathode.

When currents of over 10 amperes are to be used the form of cell shown in Figs. 97, 98 is preferable. An insulating rim — Form of Voltameter for Larger Currents.

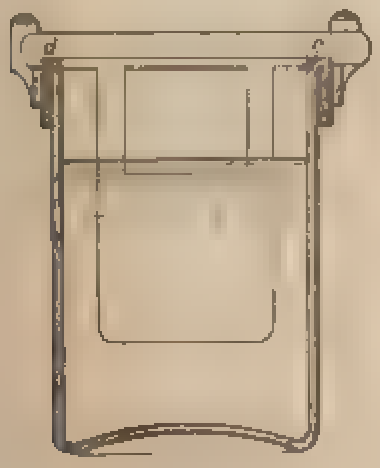


Fig. 97.

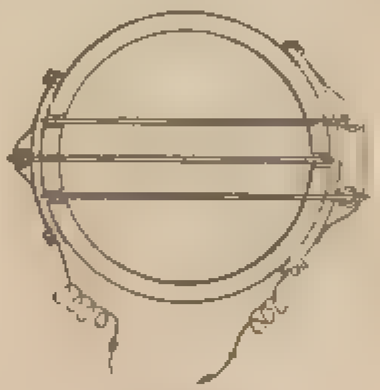


Fig. 98.

rests on the top of the cell, which for the larger sizes is conveniently made of earthenware and of rectangular shape. A groove in the rim fits the top of the cell loosely so that the rim with its attachments can be easily removed and cleaned. To the rim are fixed on opposite sides two sets of spring clips, each

made as shown in Fig. 99, by soldering flat strips of springy metal to a stiff base-piece which can be screwed to the insulating rim of the cell. To make the effective area of the plates as great as possible in comparison with the ineffective part, the part above the liquid is cut away to two narrow strips connecting the lower part to an upper cross bar c, d. One end c of this cross-bar rests in a clip, the other in a notch in the insulating rim. Anode plates and kathode plates alternate with one another, and there is one more of anodes than of kathodes, so that each kathode is between two anodes. In large cells where the plates are close and liable to touch, they are kept apart by two U-shaped glass tubes hung over each alternate plate.

FIG. 99.

Preparation of Plates.

With regard to the size and preparation of plates it was found that in the cases of both silver and copper there is a certain density of current (current strength per unit of area of plate) which gives the most adherent and, in the case of silver, most finely crystalline deposit. When silver is used there is a tendency, if the plate be too large or too small, for the crystals of deposited silver to grow out branch-like from one plate to the other, an effect which is most marked where there is a sharp edge or corner. Hence the plates must have their edges and corners rounded off to prevent the formation of these "trees," which cause great risk of loss of silver from the plate in its treatment before being weighed.

Electrolysis of Silver.

The best deposit has been found to be obtained with a solution made with five parts by weight of nitrate of silver to 95 of water, and a kathode plate giving not more than 600 sq. cms. nor less than 200 sq. cms. of active face to the ampere of current. If a stronger solution be used, the density of current may be somewhat increased, but the strength should not be less than 4 per cent. nor greater than 10 per cent.

The anode plates should be considerably greater in area than the kathode plates if their surface is to remain bright and moderately hard so as to admit of the plates being weighed if necessary. The density of the current for them should be less than one ampere to 400 sq. cms.

If the anodes are of rolled sheet silver the surface skin should be polished off with fine silver sand, and the plate washed in distilled water before being used, as otherwise the silver would be dissolved away from under the skin, which would hang as a loose sheet ready to break away when the plate was moved. A plate of silver becomes soft and inelastic by repeated use as an anode, owing to solvent action going on

below the surface, and to remedy this should be heated after being used each time to a red heat in the flame of a spirit lamp.

Treatment of Silver Plates.

The following mode of treating silver plates has been found very successful. The plate cut from the new sheet has its corners first rounded and smoothed, then is polished with fine silver sand in water, rubbed on with a soft clean pad of cloth, so as to remove the skin above referred to, and still leave a smooth surface. A gentle stream of clean water is then run over the surface from a tap to remove the sand, next the plate is washed, first with clean soap and water, then with water alone, then immersed for a few minutes in a boiling solution of cyanide of potassium, and finally washed thoroughly in a stream of clean water. The plate is dried in a current of hot air, for example before a clear fire; and great care must be taken in handling it after it has been cleaned not to touch it with the fingers, otherwise the parts which have been in contact with the skin will receive no deposit. Of course the plate must be allowed to cool before it is weighed to obviate risk of disturbance from air currents in the balance case.

When the silver deposit is to be washed and weighed, the plates are gently removed by easing the springs to prevent risk of rubbing off metal by the friction of the clips, then dipped gently in clean, recently distilled water contained in a glass vessel, so that any small crystals which may fall from the plate may be detected. The adherent nitrate solution is thus to a great extent removed, and the plates are then laid in the bottom of a shallow glass tray containing clean distilled water, and washed by gently tilting one side then the other of the tray so as to make the water flow gently over their surfaces. Then they are washed in a second tray in the same way, and allowed to soak for a quarter of an hour before being dried.

To dry the plates one corner is laid on a pad of blotting-paper and the greater part of the water drained off. The plate is then dried by holding the upper end in a spirit flame.

Electrolysis of Copper Sulphate.

The electrolysis of copper sulphate with copper anode and kathode gives results which for very high accuracy in standardizing are but little if any inferior to those obtained with silver: for most practical purposes results quite accurate enough can be obtained with much less experimental skill on the part of the operator. Repeated experiments made in the Physical Laboratory of the University of Glasgow, in connection with the graduation of Sir W. Thomson's standard instruments,* have

* See the Ref. p. 417 above. The remarkable concordance of standardizings made at different times is illustrated by results quoted in Mr. Meikle's paper.

shown that under certain easily fulfilled conditions the method of standardizing by the electrolysis of copper sulphate is perfectly accurate and trustworthy.

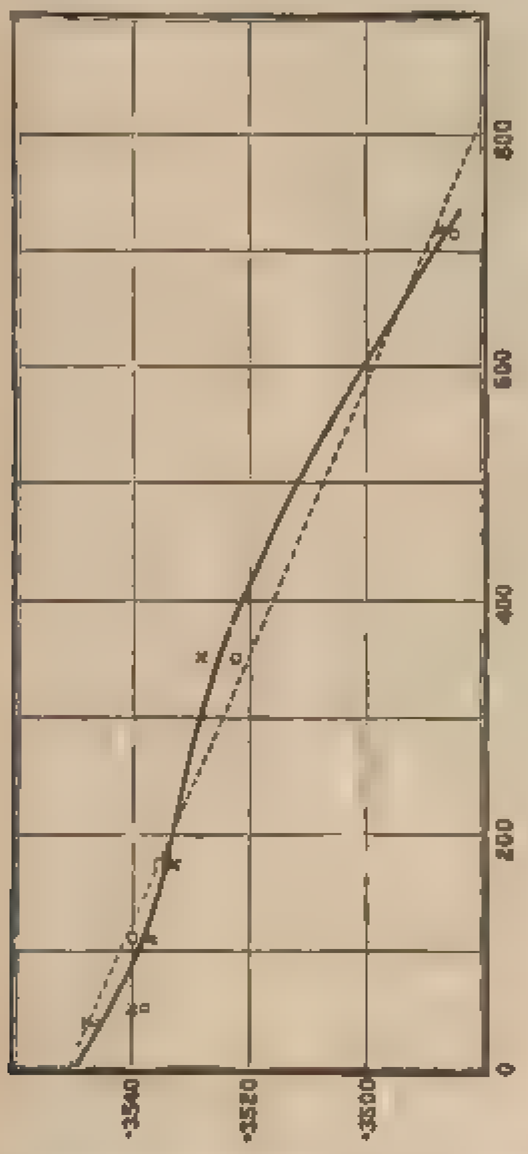
Relation of Size of Plate to Current Strength. The size of plates is not of so great importance as in the case of silver, but the kathode plate for the best results in long-continued electrolyses should have about 50 cms. of active surface or upwards per ampere. When the current is of small density deposits are obtained which are much more solid and adherent than those of silver, and therefore much more easily dealt with. As in the case of silver the anode should be of much greater area than the opposed surface of the kathode. With a density of current of upwards of $\frac{1}{50}$ of an ampere per sq. cm. the resistance at the anode becomes variable and very considerable, sometimes almost stopping the current, which after a little, with evolution of gas at the anode, regains nearly its former strength.

Effect of State of Solution as to Acidity. It was found by Prof. T. Gray in the experiments above referred to that satisfactory and concordant results could be obtained with a solution of any ordinary pure commercial copper sulphate made with pure water, provided the density did not fall below 1·05, and the solutions were made slightly more acid than in their normal state. An addition for example of $\frac{1}{10}$ per cent. of sulphuric acid to different solutions, which gave results differing among themselves, brought them into complete accordance. The loss of weight which is well known to take place when a copper plate is left standing in a copper sulphate solution, was also carefully investigated.

Loss of Copper Plate Standing in Copper Sulphate. This loss it was found seldom exceeds $\frac{1}{200}$ of a milligramme per sq. cm. per hour, or about $\frac{1}{3000}$ of that which would be deposited by a current of one ampere per 50 sq. cms. When the current density is smaller than this the loss is nearly the same as when no current flows. The effect seemed to have a minimum for a density of solution between 1·10 and 1·15, and seemed for this density to be rather retarded than the reverse by the addition of a small percentage of free acid.

Treatment of Copper Plates. The kathode plate having been cut and rounded at the corners is polished with silver sand in the same manner as the silver plate. It is then placed in the cell and a thin coating of copper deposited over it, while the current (if a large current is to be used) is adjusted to its proper strength by placing resistance in the circuit. The plate is then removed, washed in clean water and dried before a clear fire without being sensibly heated. Any defect in the first cleaning will be shown by the deposit, and if no such defect is shown, the plate is weighed and replaced in the cell for the continuation of the electrolysis. If feeble

currents are to be used this preliminary adjustment is hardly necessary, as it is preferable then to use a larger number of cells than are absolutely necessary to produce the current, and bring down the current to the necessary strength by adding an amount of resistance which can be easily enough estimated



Abscissæ give area of kathode in square centims per ampere. Ordinates are electro-chemical equivalent, multiplied by 10000

FIG. 100

After the electrolysis the plates are carefully removed and at once dipped in ordinary (not necessarily distilled) clean water, containing two or three drops of sulphuric acid per litre, then washed in a tray like the silver plates The plates are then rinsed in clean water without acid, and dried first in a clean pad of white blotting paper, and then before a fire or over a spirit

lamp. If this is carefully done and the deposit be fairly good no copper will be lost and there will be no gain of weight by oxidation. The plates may be weighed after having been allowed to cool down to the ordinary temperature

The anode plates are treated in a similar manner (except as regards the drying in a blotting pad, which might cause loss of silver) without loss of copper, or gain by oxidation, but owing to loss of weight in the solution &c., they give much less satisfactory results than do the kathode plates.

The arrangement of the circuit for electrolytic experiments consists of a battery of tray Daniells, or other constant cells, joined in series with the electrolytic cells to be used, a sensitive galvanometer, and a rheostat (or other readily variable resistance) by which the current is to be regulated. The current is adjusted so that a convenient deflection is obtained, which is restored by slightly turning the rheostat in the proper direction, if any alteration takes place. The conduct of an experiment will be understood from the description of the process of standardizing given below.

Electro-Chemical Equivalent of Copper.

From Lord Rayleigh's result for the electro-chemical equivalent of silver (see p. 416 above), namely that a coulomb deposits ·0011179 gramme of silver, very nearly, Professor T. Gray has determined by comparison the electro-chemical equivalent of copper, and found it to be very approximately ·0003287 (or for practical purposes ·0003290) at ordinary temperatures, and with a current density of one ampere per 50 sq. cms. of active surface of kathode. This number can be corrected for other current densities by the dotted curve given in Fig. 100.

The results from which this curve has been plotted are given in the following Table.

Electro-Chemical Equivalents for Different Current Densities.

AMOUNTS OF COPPER DEPOSITED BY THE SAME QUANTITY OF ELECTRICITY ON KATHODE PLATES OF DIFFERENT AREAS.

Area of plate in sq. cms.	Amount of deposit in grammes (first experiment).	Amount of deposit in grammes (second experiment).
3	·3534	·3534
5	·3530	·3529
11	·3528	·3530
18·5	·3526	·3527
36	·3524	·3521
73	·3503	·3502

The effect of variation of temperature * on the amount of copper deposited has been found by Mr A. W. Meikle to be very slight at ordinary temperatures; for a change from 12° C. to 28° C. it is a diminution for a given size of plate of only $\frac{1}{10}$ per cent.

At temperatures rising above 20° C. the effect of variation of size of plate becomes more and more important.

Graduation of Standard Instruments by Electrolysis.

The application of electrolysis to the standardizing of instruments will now be illustrated by a short account of its application to the determination of the proper weights for use in Sir William Thomson's standard current balances described above. The

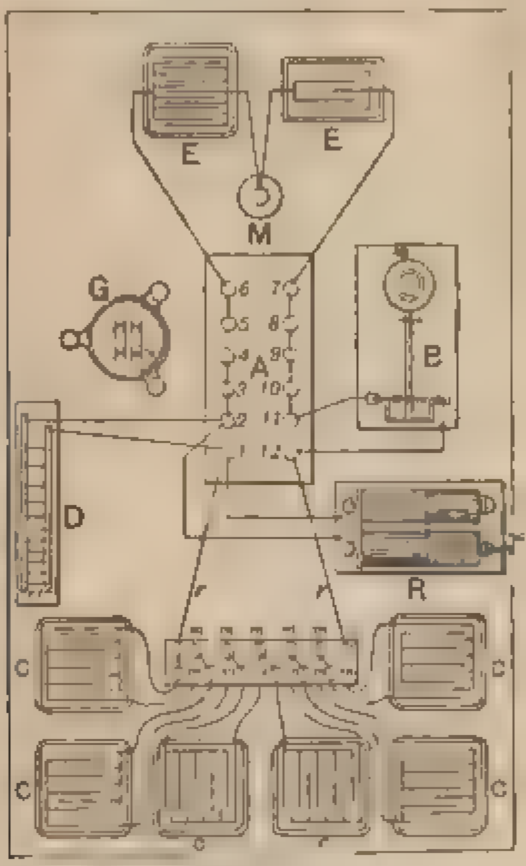


Fig. 101.

arrangement of apparatus is shown in Fig. 101, which may be taken as a plan of the standardizing table with instruments in position. C C C C C C are six of the Electric Power Storage Co.'s secondary cells, shown joined in series, by being connected

* See Ref. p. 417 above.

to a series of mercury cups, $m, m,$ . . . which are connected across by thick copper rods as indicated by the full and dotted lines. (These cups are on a vulcanite base, and have bottoms of thick copper to ensure contact.) When however currents of great strength are required for the graduation of low resistance instruments, these cups are joined in parallel by two rods of copper which have teeth at the proper distance apart to fit into the cups, so as to join all in each row together. The battery fully charged and thus joined in parallel will maintain a current of 200 amperes for 10 hours.

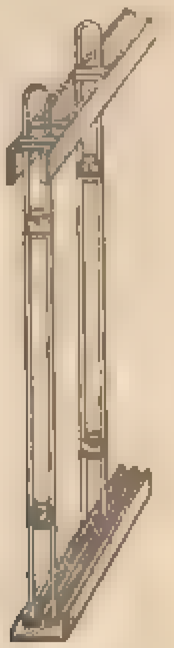


FIG. 102.

Arrangement of Apparatus. The terminal cups of the commutating board are shown joined to a distributing board provided with cups, 1, 2, . . . 12, by which the battery is put in series with a rheostat R, in parallel arc with a set of conductance bars D, a galvanometer G, a pair of large electrolytic cells joined by a movable cup M, and finally the balance B to be standardized. The conductance bars

Conductance Bars. are constructed as shown in Fig. 102. Rods of platinoid of thickness according to the conductance required are bent into U-shape as shown, and the limbs held at proper distances apart by wooden blocks at intervals, or by a strip of wood running along their whole length, according as the rods are thick or thin. The length of rod in each U is about 4 metres, and the thickness is chosen such that one or two volts difference of potential

produces very little heating of the wire. The troughs, *tt* (Fig. 98), are made with bottoms of thick copper and contain mercury in which the ends of the rods (or thick copper pieces soldered to the wires if thin) rest pressed down by their own weight. The different *l*'s beginning from one side are graduated so as to have conductances nearly in the ratios 1 : 1 : 2 : 4, &c., so that the total conductance in the set may be increased at will by a step equal to the lowest conductance (since each conductance is that amount greater than the sum of all that precede it in the series). When any bar is not in use its lower ends are lifted out of the troughs as shown in the Figure. The rheostat, which has a least conductance rather less than that of the smallest bar, furnishes an auxiliary variable bar by which the conductance can be gradually altered. Its wire is of stranded copper and can carry 10 amperes without damage.

Determination of Constant of Balance.

The current balance has previously had its scale graduated and attached as described above, and it remains only to show how the constant of the instrument is determined, or in other words the weight which placed on the beam will enable the current to be obtained from its indications in the manner already described (p. 409). A chosen arbitrary counterpoise weight is placed in the trough, and another, which then just brings the beam to the sighted position without current when at the zero of the scale, is placed on the beam with the index at some division near the right-hand end so that a current of, say, 10 amperes (more or less according to the instrument) is required to bring the beam to the sighted position. The electrolytic cells are then arranged to give about 500 sq. cms. of kathode surface, and are joined up with a conductance sufficient to give nearly the required current. The balance will come nearly to zero, and is brought to zero exactly by adjusting the current by means of the rheostat. These adjustments having been made, the kathode plates are removed, washed, weighed, and replaced. At an instant observed on an accurate time keeper the circuit is closed, and any deviation of the current corrected by means of the rheostat. The current is brought to its correct value in from five to ten seconds, and hence in an electrolysis of say an hour (the usual duration of an experiment) the error due to its deviation from the final constant value for this short variable period is quite imperceptible. Any variations of the current strength are observed on the instrument itself, or if (which rarely happens) that is not sensitive enough, on a more sensitive galvanometer *G* (Fig. 101), which is introduced when required, and kept out of circuit at other times. Any sufficiently sensitive instrument which can have its (not necessarily known) constant changed by

Procedure in Electrolysis.

any required amount by varying the field at the needle, or by using an instrument provided with two parallel coils with the needle midway between them, and arranged to permit the distance of the coils apart to be altered at pleasure, is convenient for this purpose.

Calculation of Beam and Counterpoise Weights.

The electrolysis having thus been carried on and completed, the circuit is broken, and the plates washed and weighed. The current is calculated from the result by dividing the gain of copper on the kathode expressed in grammes, by the electrochemical equivalent of copper (·0003287, or, as explained above, the proper value for the density of current), and the result by the number of seconds during which the electrolysis has lasted. Let C be the current for the position of the weight on the beam as given by the table of doubled square roots, w_1, w_2, the corresponding counterpoise and beam weights respectively, C' the current given by the electrolysis, w'_1, w'_2 the counterpoise weight and beam weight applied, then we have

$$\frac{C^2}{C'^2} = \frac{w_1 d_1 + w_2 d_2}{w'_1 d_1 + w'_2 d_2}$$

where d_1, d_2 are constants. But $w_1/w_2 = w'_1/w'_2$; hence this equation gives

$$\frac{C^2}{C'^2} = \frac{w_1}{w'_1} = \frac{w_2}{w'_2}.$$

Thus w_1, w_2 are found by multiplying the ratio C^2/C'^2 by w'_1, w'_2 respectively, and the determination is complete.

Arrangement for Strong or Weak Currents.

When a very strong or a very weak current is required, as in the graduation of a hektoampere or a centiampere balance, it is desirable in the former case to allow the whole current to flow through the instrument, and only a convenient part through the electrolytic cell, and in the latter case to use a considerably greater current through the electrolytic cell than through the instrument. The current must therefore be divided in both these cases into two parts whose ratio is accurately known, and this may be done by the conductance bridge shown in Fig. 100. A set of parallel straight wires of platinoid are each soldered at one end to a thick terminal bar of copper b, and have soldered to them at the other ends thick terminal pieces of copper by which they can be connected in two groups by means of mercury troughs b_1, b_2. In the figure they are shown in two groups of 10 and 1 respectively.

The wires are adjusted so that when they are in position they

have all precisely the same resistance. Between the troughs b_1, b_2, a sensitive reflecting galvanometer (see Vol. I. p. 307, g is joined which indicates no current when b_1, b_2 are at the same potential. The electrolytic cells E, E', and the instrument G to be standardized, are placed as shown in the figure when the

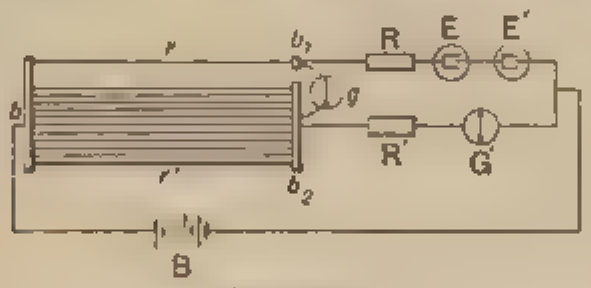


FIG. 103.

standardizing current must be greater than that which the cells can carry, and the positions shown are interchanged when the reverse is the case. The currents are adjusted to balance in both cases by the rheostats R, R'. The currents are of course in the ratio of the conductances of the groups r, r' of the wires of the bridge.

SECTION III

DETERMINATION OF ELECTROMOTIVE FORCES OF CELLS AND GRADUATION OF VOLTMETERS

When a current known in absolute measure flows through a known resistance the difference of potential between the terminals of the resistance is also known. By means of this known difference of potential, which may be varied at pleasure, a voltmeter may be graduated. A voltmeter of any type is an instrument, the resistance of which is so high that the attachment of its terminals to two points in a conductor carrying a current does not perceptibly change the difference of potential formerly existing between these points. Of course every absolute galvanometer, electrodynamometer, or standard balance measures differences of potential, for, if its resistance is known, the difference of potential between its terminals can be calculated

Potential Measuring Instruments or Voltmeters.

from Ohm's law; but the convenience of a voltmeter especially made with a high resistance coil is that its terminals may be applied at any two points in a working circuit, and the difference of potential, thus calculated as existing between these two points while the terminals are in contact, may, in most cases, be taken as the actual difference of potential which exists between the same points when nothing but the ordinary conductor connects them. For, let V be this actual difference of potential in volts, let r ohms be the equivalent resistance of the whole circuit between the two points and R ohms the resistance of the voltmeter. Then (Vol. I. p. 161) by the application of R, V is diminished in the ratio of R to $R+r$, and therefore the difference of potential between the ends of the coil is now $VR/(R+r)$. Hence by Ohm's law we have for the current through the galvanometer the value $V/R(1+r/R)$. If r be only a small fraction of R, r/R is inappreciable, and the difference of potential calculated from the equation $C = V/R$ will be nearly enough the true value. So far, it is to be observed, r is the equivalent resistance between the two points, and the result stated holds, however the electromotive force may have its seat in the circuit, if only R be great in comparison with r. If, however, either of the two parts of the circuit between the two points in question have a resistance r' small in comparison with R, then, as can be easily proved, the value of the difference of potential between the terminals of r is practically unchanged by the addition of R as a derived circuit.

Graduation of Voltmeter. The voltmeter has its terminals attached to those of the resistance through which the current is flowing; or, if the standard measuring instrument is sensitive enough, the measured current is sent through the voltmeter itself; and readings of needle or other indicator are taken. In either case the readings are proportional to the difference of potential between the terminals of the instrument, but in the former arrangement the difference of potential is equal, in volts, to the current in amperes flowing through the resistance multiplied by the value of the resistance in ohms, in the latter the difference of potential is equal to the measured current through the voltmeter into the resistance between its terminals.

If the scale of the instrument does not follow any known law, it is necessary to determine by direct experiment the electromotive force corresponding to different deflections and thus, so to speak, calibrate the instrument. To do this the most convenient plan is to divide the scale accurately into equal divisions and to number these from zero at the position of equilibrium with no current. Then the current measured by the standard

galvanometer is varied conveniently by introducing resistance into the circuit by a rheostat, and the deflection observed for several different values. The corresponding differences of potential are then plotted on squared paper as ordinates for which the number of divisions of the deflections are the corresponding abscissæ. A curve is then carefully drawn through the extremities of these ordinates, and the ordinate of this curve drawn for any chosen abscissa will be the difference of potential for that deflection.

For verifying the accuracy of the graduation of the potential instruments when performed by either of these methods, or for actually performing the graduation when other methods are not convenient, some form of voltaic cell of known electromotive force may be used.

Clark's Standard Cell

As the result of many careful experiments made by Lord Rayleigh and others, it has been found that the most reliable standard cell is that proposed by Mr. Latimer Clark. When certain precautions are taken in its preparation the electromotive forces of different specimens are very nearly the same, and remain constant for a long time provided care is taken to prevent more than a very feeble current from ever passing through them.

Mode of Setting up Clark Cells.

The cell may be made in a reliable and handy form in the following way, which includes the precautions that Lord Rayleigh's experience * has shown to be necessary. The vessel is a weighing tube, or for small sizes merely a test-tube, with a platinum wire sealed through the bottom, and rests on a suitable stand as shown (Fig. 104). This wire makes contact with mercury, which occupies the bottom of the cell and forms one of the plates. The mercury must be pure, and it is desirable to ensure its being so by redistilling in vacuo good clean commercial mercury. On the mercury rests a paste made by adding to 150 grammes of mercurous sulphate 5 grammes of zinc carbonate, and sufficient saturated zinc sulphate solution to give a stiff pasty consistency.

The zinc sulphate solution should be made from pure zinc sulphate dissolved under gentle heat in distilled water so as to

* See Lord Rayleigh and Mrs. Sidgwick's paper *On the Electro-Chemical Equivalent of Silver* already cited (*Phil. Trans.* Part II. 1884), also Lord Rayleigh on the *Clark Cell as a Standard of Electromotive Force* (*Phil. Trans.* Part II. 1885). These papers contain particulars of the method of determining the electromotive force of Clark Cells, and the latter especially details of the mode of constructing them, of which an abstract is given below in the text.

make a saturated solution, and, after having been allowed to stand for some time to precipitate any iron which may have been present in the sulphate, filtered in a warm place into a stock bottle. When required the solution is gently warmed, and drawn off by a siphon from just above the crystals at the bottom. The paste is made by placing the mercurous sulphate and zinc carbonate in a mortar and rubbing in the zinc sulphate at intervals during two or three days, to give time for all carbonic acid to pass off.

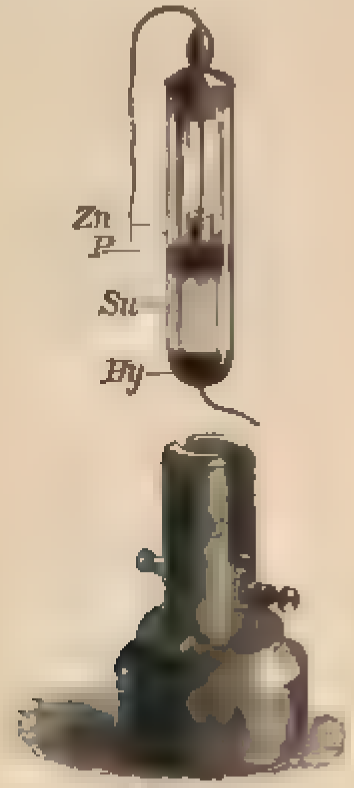


FIG. 104

A rod of what is called "redistilled zinc" resting in the paste, and held upright in the vessel by a notched ring of cork, forms the other plate. The zinc is cleaned before putting it in position by dipping it in sulphuric acid and then washing it in distilled water. Connection with it is made by a gutta-percha-covered copper wire soldered to it, and passed up through a cork which closes the cell and nearly fills the upper part of it so that very little air is included. The cork is flush with the top of the tube, and the edges of the tube and the whole upper surface of the cork is covered with marine glue to seal up the cell.

A cell thus made, if used with only the feeblest currents, never

E.M.F. of Clark Cell.

short-circuited, nor exposed to great variations of temperature will have a constant electromotive force E in volts at temperature t C. given according to Lord Rayleigh and Mrs. Sidgwick's determination (if 1 B.A. unit = ·9866 ohm) by the equation

$$E = 1{\cdot}4345\{1 - {\cdot}00077\,(t - 15)\} \quad . \quad . \quad . \quad . \quad (64)$$

Determination of E.M.F. of Clark Cell.

The method employed by Lord Rayleigh and Mrs. Sidgwick in the determination of the electromotive force of the Clark cell, and the method of using the cell for purposes of graduation, will be understood from Fig. 93. (For convenience Fig. 93 is here repeated.) Two Leclanché cells M, and two resistance

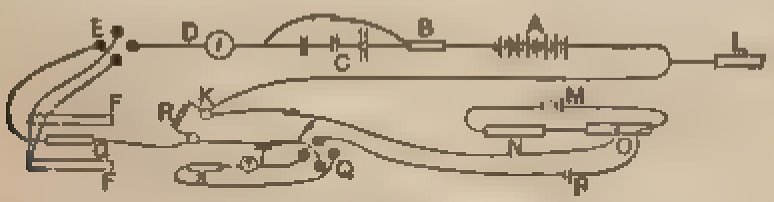


Fig. 105.

boxes N, O, were joined in circuit. At two points in O were attached two wires in one of which was placed the Clark cell P, which was to be tested. These wires formed with a resistance R a derived branch of the circuit of M including a mercury reversing key Q, a Thomson reflecting galvanometer T, and a resistance S of 1000 ohms.

In the earlier experiments the galvanometer had in its coils a resistance of about 200 ohms, but in later determinations it was provided with a coil containing a much greater length of wire, so that a higher sensibility was obtained.

The other arrangements connected with the circuit are the battery A, the voltameters, and the current weigher as described above (p. 398). One extremity of A was connected to earth at L.

Poggendorff's Compensation Method.

The main current from A after traversing the voltameters and current weigher passes through the resistance R back to A. To prevent undue heating by the electrolysing current, which was about $\frac{1}{3}$ ampere, the resistance R was constructed of bare german silver wire wrapped round a frame of two parallel ebonite rods kept apart by wooden bars, and was provided with stout copper terminals which rested on the copper bottoms of cups H, K filled with mercury. The resistance R was 4·00699 B.A. units at 17·6 C. This was corrected for the difference between 17·6 C. and the temperature of the atmosphere, and also for heating pro-

duced by the current. It was found that a correcting factor 1·00041 had to be applied to take account of the latter effect.

Mode of Adjustment to Balance

In the first determinations the battery M was not used and the electromotive force of P was balanced by the difference of potential existing between its terminals H and K. The adjustment to balance was made by placing a high resistance box in multiple arc across R, between H and K, and unplugging resistance until with the current flowing through the voltameters, no current passed through T when the derived circuit was thrown in for a moment.

The difference of potential between H and K was then obtained from the resistance of the double arc now constituting R, and the absolute value of the current given by the electrolysis. The value of the current at the instant when P was balanced could be obtained from the curves (Fig 91) showing the results of the two current weighings; and thus several determinations of electromotive force could be made in a short time.

In later determinations the balance was finally adjusted by including in the derived circuit with P a part of the electromotive force of the pair of Leclanché cel s. An independent comparison of the electromotive force of the Leclanchés with that of the Clark cell, was made by balancing the Clark cell, in the manner just described, by the difference of potential between two points of a resistance in circuit with the Leclanchés. This enabled the part of the balancing electromotive force supplied by the Leclanchés to be found from the known resistance intercepted between the terminals of the derived circuit and the whole resistance in N and O together, which was kept at 10,000 ohms.

The following values have been obtained by other experimenters for the electromotive force of a Clark cell at 15° C.:—

Carhart	1·434 volts
Kahle (Zeitschrift für Instrumentenkunde).	1·4341 ,,
Glazebrook and Skinner, Proc. R S. 54 (1892)	1·4342 ,,

Graduation of Voltmeter by Standard Cells.

Standard cells of known electromotive force being available they may be used for the graduation of voltmeters by the same compensation method. A circuit is made of a battery A (Fig 106) of storage or Daniell's cells, in series with resistances R, S, and the voltmeter G to be graduated A battery of a suitable number of standard cel s has its terminals applied at the extremities of the variable resistance R, and its circuit contains a sensitive galvanometer D, and a key K. R or S is adjusted until no current flows through D when the key K is tapped down for an instant. When this is the case the electromotive force of C is balanced by the difference of potential at the two ends of R pro-

duced by B. Hence the difference of potential in volts then existing between the terminals of G is given (for a Clark's cell at 15° C.) by the equation

$$V = 1{\cdot}4345 \frac{G}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

By this method, which is an application of Poggendorff's method of comparing the electromotive forces of batteries, balance is obtained when no current is flowing through the standard cell, and disturbance from polarization is altogether avoided. It has been found very easy and convenient in practice

Raoult's Standard Daniell.

Some form of Daniell's cell is easily set up, and though less reliable is convenient for use as a standard when Clark cells are not available. A well-known form is Raoult's, which has the zinc

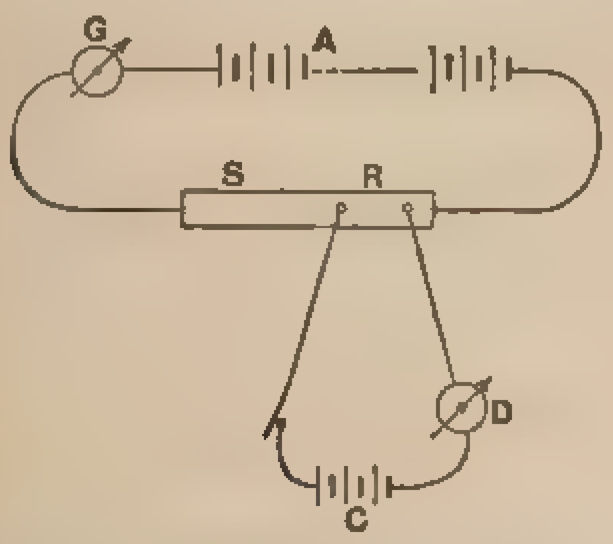


FIG. 106.

and copper solutions in separate vessels connected by a tube filled with zinc sulphate and tied over the ends with bladder This, when made with a plate of pure zinc amalgamated with mercury and a plate of electrolytically deposited copper, was found by Lord Rayleigh to have an electromotive force of approximately ·7703 of that of a Clark cell.

Thomson's Standard Daniell.

A standard Daniell's cell has been proposed by Sir W. Thomson, which consists of a zinc plate resting at the bottom of a glass vessel in a stratum of saturated zinc sulphate, and a copper plate in a solution of copper sulphate of density 1·02, which has been so gently formed in the stratum of zinc sulphate as to leave

a clear surface of separation. The copper sulphate solution is introduced by means of a glass tube dipping down into the liquid and terminating in a fine point, which is bent into a horizontal direction so as to deliver the liquid with as little disturbance as possible. This tube is connected by a piece of india-rubber tubing with a funnel, by the raising or lowering of which the sulphate of copper can be run into or run out of the cell. By this means the sulphate of copper is run in when the cell is to be used, and at once removed when the cell is no longer wanted. The solutions should be kept in stock bottles and the cell set up fresh when wanted.

Method of Using Standard Daniell.

The standard Daniell's cell is very conveniently used along with a Daniell's battery in the manner represented in the diagram (Fig. 107). C is the standard cell, and B a battery of

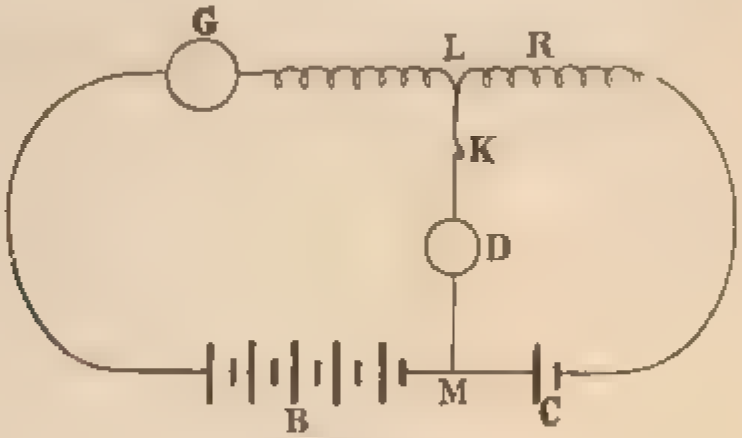


Fig. 107.

from 30 to 40 small gravity Daniells.* A circuit is formed of a resistance box, the galvanometer G to be graduated, and the battery B joined in series with the standard cell C. A sensitive galvanometer D, which may be a reflecting galvanometer, or any

* These can be very easily made by using large preserve pots as containing vessels, and placing at the bottom of each a copper disc of from three to three and a half inches in diameter in a stratum of saturated copper sulphate solution, and a grating or plate of zinc a little below the mouth of the vessel immersed in a solution of zinc sulphate, of density 1·2. The copper sulphate may be kept saturated by crystals dropped into a glass tube passing down through a hole in the zinc plate to the copper. A copper wire well covered with gutta percha should be used as the electrode of the copper plate.

very sensitive galvanometer of low resistance, has one terminal attached at a point M between the battery and the standard cell, and the other terminal through the key K to an intermediate terminal L of the resistance box. The resistances in the box, on the two sides of L are adjusted until no current flows through the galvanometer D, when the key K is depressed.

Let R be the resistance in the box to the right of L, r the resistance of the cell C, and G the resistance of the galvanometer. Then if V be the difference of potential, in volts, between the terminals of the galvanometer, E. M. F. of Daniell's Cell.

$$V = 1{\cdot}072 \frac{G}{R + r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

In practice a resistance of from 300 to 400 ohms is generally required for R. The electromotive force of the standard cell was determined by Prof. T. Gray and found to be 1·072 volts at ordinary temperature. A determination of the electromotive force of the same cell has also been made by Lord Rayleigh, who found it to be ·743 of a Clark cell, the electromotive force of which was 1·4542 B.A. volts, nearly, at 15°. This would give very approximately 1·068 true volts for the Daniell cell. It was taken as 1·072 volts, as, notwithstanding the large battery in the circuit, the total resistance is so great that there is very little polarization. This method in fact is peculiarly well adapted for the Daniell's cell, as the slight current flowing through serves to keep its plates in a constant and fresh state. It is known as Lumsden's and also as Bosscha's method of comparing electromotive forces.

The difference of potential, the magnitude of which is thus obtained, is chosen such as to give a convenient deflection on the instrument to be graduated.*

* For further particulars regarding the graduation of instruments, see the Author's smaller Treatise.

CHAPTER VIII

MEASUREMENT OF INDUCTANCES

Comparison of Induction-Coefficients or Inductances.

THE subject of the experimental determination of coefficients of induction, or, as they are now called, *inductances*, with one another, with known resistances, and with electrostatic capacities, has attracted much attention during the last fifteen years. This is a consequence mainly, on the one hand, of the efforts that have been made to obtain a more accurate determination of the ohm, and of the ratio of the electromagnetic to the electrostatic unit of quantity of electricity; and on the other of the vastly increased importance given to induction in electrical theory and practice by the enormous development which has lately taken place in the use of dynamos and especially of alternate-current machines.

In what follows an attempt is made to describe the various methods of comparison which have been devised, giving in each case a fairly full account of the theory of the method, and, if possible, an illustration of the solution obtained by one or more examples of actual experiments. We shall use Mr. Oliver Heaviside's term "inductance" to signify what is generally denoted by "coefficient of induction," distinguishing where necessary between *mutual inductance* and *self-induct-*

ance; but as self-induction is relatively more important, and is much more frequently referred to than mutual induction, we shall, where no ambiguity is likely to arise, use the single word "inductance" in the sense of "coefficient of self-induction."

It is convenient to consider in the first place some points of general theory which are of importance in this connection. The equations of varying currents in any conductor, or circuit of conductors, are obtainable from the electrokinetic energy and the dissipation function, when these are known, if only electrokinetic phenomena are in question, or from these two expressions, together with that of the electrostatic energy, if, as will be the case in some of the problems in the present Chapter, electrostatic phenomena have also to be taken into account.

General Theory of Network of Conductors. Carrying Varying Currents.

Equations of currents have been obtained above (p. 160) by considering an assemblage of complete circuits as a dynamical system; but similar equations are obtainable in precisely the same way for the currents in the individual conductors of a network, provided that instead of resistances, inductances and electromotive forces in circuits, the resistances, inductances, self and mutual, of the conductors, and the impressed differences of potentials between their terminals are used. The only difficulty is as to the meaning of the self-inductance of a conductor joining two points in a circuit, or the mutual inductance of two such conductors in the same or different circuits. But all such questions are resolved by adopting some proper mode of calculating inductance [for example Neumann's formula pp 129, 171] which enables the inductance of a conductor to be found as that of a part of a circuit, in the sense that when the inductances of the parts are calculated in this way they give the proper value of the electro-kinetic energy of the circuit or circuits for any possible arrangement of currents. A case in point is that of two or more *coils* joined in multiple arc between two points *AB*. The inductances for these conductors are very approximately those obtained by regarding the coils as made up of so many complete circuits given in dimensions and position by the turns of wire. In such cases the flux of magnetic induction through the part of the circuit considered is definite and calculable, and different methods lead to the same result. But there are other cases, for example that of a Hertzian vibrator, in which different processes lead to distinctly different values of the self-inductance of the apparatus (see Chap. XIV. below).

Self or Mutual Inductance of part of a Circuit.

Maxwell's Cycle-Method of a Network. The difficulty here indicated is avoided by a device adopted by Maxwell. A network is made up of a series of meshes or "cells," in which each individual conductor, except those forming the outer edge of the network, is common to two meshes. Maxwell supposed a current to circulate round each mesh in the same direction, so that the actual current in each conductor was the difference of the currents round two adjoining meshes. Thus each mesh is a closed circuit with its own current in it, and the self and mutual inductances of the system are perfectly definite, being those due to the various closed circuits each supposed to carry unit current.

Electrokinetic and Electrostatic Energies and Dissipation Function. Taking the former method first let L_1, L_2, . . denote the self-inductances of the different homogeneous conductors of the system supposed linear $\dot{y}_1$, $\dot{y}_2$, ... , the currents in them, M_{12}, M_{23}, ... the mutual inductances of the conductors indicated by the suffixes, then as at p 185 the electrokinetic energy is given by

$$T=\tfrac{1}{2}\left(L_1\dot{y}_1^2+2M_{12}\dot{y}_1\dot{y}_2+\ldots+L_2\dot{y}_2^2+2M_{23}\dot{y}_2\dot{y}_3+\ldots\right) \quad . \quad (1)$$

The dissipation function is

$$F=\tfrac{1}{2}\,\Sigma R_k\dot{y}_k^2 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

where R_k denotes the resistance of the conductor in which the current is $\dot{y}_k$. If E be the electrostatic energy due to the charge of condensers

$$\mathsf{E}=\tfrac{1}{2}\,\Sigma C_m V_m \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

where C_m is the capacity of a typical condenser of the system charged to a difference of potential V_m between its coating.

The effect of the electrostatic capacity of the conductors concerned is something quite sensible, and may in certain cases be allowed for When it can be expressed, the part of the electrostatic energy which depends on the capacity of the conductors, enables the terms in (6) below to be calculated.

By this expression, also, when it can be calculated for the different parts of the conductors, the electrostatic capacities of the connecting wires can be taken into account. In such cases, however, the capacity can in general only be roughly estimated.

Equation of Current in a Single Conductor. Bringing then into the account the electrostatic energy regarded as potential energy, we have to add to the electrokinetic applied force corresponding to the current $\dot{y}_k$ the electrostatic force $-\partial\mathsf{E}/\partial y_k$. Thus if V_k denote the difference of potential

between the terminals of the conductor in which the current is $\dot{y}_k$ the typical equation of current is

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}_k}+\frac{\partial F}{\partial \dot{y}_k}=V_k-\frac{\partial \mathrm{E}}{\partial y_k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equation of Currents in a Circuit of a Network.

Writing down the equation of this type for the successive homogeneous conductors taken in order round a circuit of a network, and adding both sides of the equations, we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_j}+\frac{\partial T}{\partial \dot{y}_{j+1}}+\ldots\right)+\frac{\partial F}{\partial \dot{y}_j}+\frac{\partial F}{\partial \dot{y}_{j+1}}$$

$$+\ldots=E-\left(\frac{\partial \mathrm{E}}{\partial y_j}+\frac{\partial \mathrm{E}}{\partial y_{j+1}}+\ldots\right) \quad . \quad (5)$$

where E is the total internal applied electromotive force in the circuit, since we know that the latter is the sum of the differences of potential between the terminals of the successive homogeneous conductors forming it. This equation may be written

$$\Sigma\,(L_j\ddot{y}_j+M_{jk}\ddot{y}_k+R_j\dot{y}_j)=E-\sum\frac{\partial \mathrm{E}}{\partial y_j}. \quad . \quad . \quad . \quad (6)$$

in which the summations are taken for all the conductors of the circuit considered.

This equation may be taken as the most general form of the so-called "second law" which Kirchhoff first explicitly stated for steady currents in a system of linear conductors. It will be of constant use in what follows.

Principle of Continuity for Varying Currents derived from Law of Magnetic Force.

The principle of continuity, commonly called Kirchhoff's first law, is generally assumed for variable currents, and it is also usual to assume, as has been done above, that at any instant the magnetic force at any point due to a varying current in a circuit is the same as would be produced by a steady current equal in intensity to that which exists in the circuit at that instant. The latter assumption is justified, for points which are *near* the circuit, by the theory, confirmed now by experiment, of propagation of electromagnetic action.

It does not seem to have been noticed that the principle of continuity for a linear circuit can be deduced from the law of magnetic force as follows. Let three wires meet at a point O, then according to the principle of continuity the rate of flow from the point must be exactly equal at any instant to the rate

of flow to the point at the same instant. Let O be taken as the centre of a small sphere, and let the wires pass through the surface of the sphere at A, B, C (Fig. 108). Let a path be drawn round the wire A on the sphere, then carried to B, then to C nearly round it, and finally back to the point of starting from A, so that a closed path is traced on the sphere, consisting of three nearly closed curves described in the same direction round A, B, C, and an infinitely nearly closed path A', B', C', not embracing any of the conductors. A magnetic pole carried round the complete path will have no work done on it on the whole, since the path does not really surround any conductor; in other words, it could be shrunk to a point, without cutting through the conductor, and the work done in carrying a pole round the infinitely nearly closed path A', B', C' also vanishes. [To see

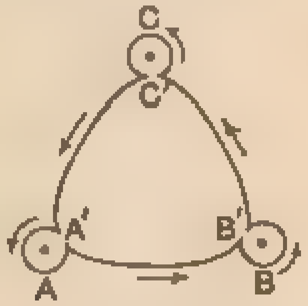


FIG. 108

this it is only necessary to conceive the path opened out into an open loop, slipped back beyond the centre of the sphere and then shrunk up.] But if γ_1, γ_2, γ_3, be the currents in the wires at A, B, C, all reckoned as inflowing, or all as outflowing, the work done on the pole in the three paths closely surrounding the wires is $4\pi(\gamma_1+\gamma_2+\gamma_3)$, and thus must be zero, since the work done round A' B' C' is zero. Hence we have

$$4\pi(\gamma_1+\gamma_2+\gamma_3)=0$$

or

$$\gamma_1+\gamma_2+\gamma_3=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

that is the total current arriving at or flowing away from the point at any instant is zero. The same thing can obviously be proved in the same way for any number of conductors meeting at a point.

Returning to the dynamical equations of currents, the equations for Maxwell's method of meshes, each carrying its own current,

Theory of Maxwell's Cycle-Method.

are easily written down. The quantities of electricity which have flowed round the different meshes from any era of reckoning up to the instant under consideration become the generalised conductors, and their time-rates of variation, or the currents at that instant, the corresponding velocities. If then L_1, L_2 ..denote the self-inductances of the different meshes, each regarded as a separate circuit, in which currents $\dot{y}_1$, $\dot{y}_2$,...flow, M_{12}, M_{23},...the mutual inductances of the pairs of meshes indicated by the suffixes, we have

$$T = \tfrac{1}{2}\,(L_1\dot{y}_1^2 + 2M_{12}\dot{y}_1\dot{y}_2 + \ldots) \quad . \quad . \quad . \quad . \quad (8)$$

Again, if R_{jk} denote the resistance of a conductor which adjoins two meshes distinguished by the suffixes j and k,

$$F = \tfrac{1}{2}\,\Sigma\, R_{jk}\,(\dot{y}_j - \dot{y}_k)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Equation of Currents.

These two equations with (3) above enable the equations of currents for the different meshes to be written down. They are thus of the type

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}_j} + \frac{\partial F}{\partial \dot{y}_j} = E_j - \frac{\partial \mathrm{E}}{\partial \dot{y}_j} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where E_j is the electromotive force in the circuit indicated by the suffix j.

Comparison of the Two Methods.

This method avoids the necessity for an explicit reference to the principle of continuity, inasmuch as this principle is assumed in the statement of the method, and it is convenient for the systematic working out of a complicated system; but with the electrokinetic energy expressed in the above form, which is the strictly accurate one when the generalised velocities are the mesh- or cycle-currents, it is not convenient for the derivation of equations from which the inductances of given conductors are to be obtained. In these applications, however, it is usual to modify the form of the electrokinetic energy by writing it

$$T = \tfrac{1}{2}\,\Sigma\,\{L_{jk}\,(\dot{y}_j - \dot{y}_k)^2 + 2M_{(jk)(lm)}\,(\dot{y}_l - \dot{y}_m)\} \quad . \quad . \quad (11)$$

where L_{jk} is the self-inductance of the conductor common to the two cycles indicated by the suffixes, and $M_{(jk)(lm)}$ the mutual inductance between that conductor, and the conductor common to the two meshes indicated by the suffixes lm. But this merely amounts to using the first method after all. In general it is quite easy to write down the equations for the different conductors from (4) for the first method, applying the principle of

continuity mentally; and as only one symbol is required for the current in each conductor, the first method has the advantage of greater brevity of expression.

Comparisons of Inductances: Problems.

The comparison of inductances comprises five problems with which we shall deal in succession: the comparison (1) of two mutual inductances, (2) of two self-inductances, (3) of a mutual inductance with a self-inductance, (4) of a mutual or self-inductance with a resistance, (5) of a mutual or self-inductance with an electrostatic capacity.

Comparison of two Mutual Inductances. Maxwell's Method.

Of the first problem the following solution has been given by Clerk Maxwell. Let A_3, A_4, (Fig. 109) be

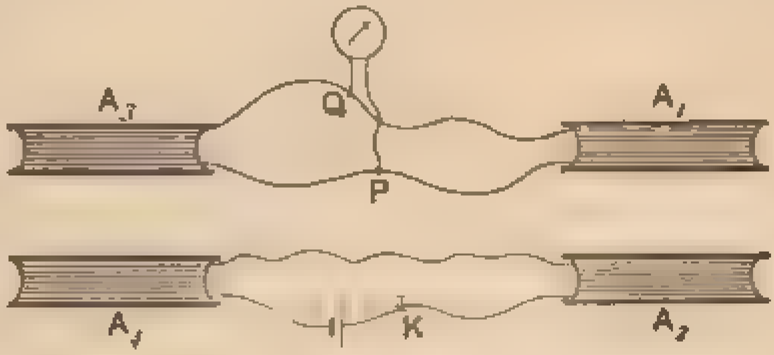


FIG. 109.

the two coils, the mutual inductance M_{34} between which is to be compared with that between two other coils A_1, A_2. A_1 and A_2, A_3 and A_4 are placed opposite one another at the required distance in the case of each pair. A circuit is made up of A_2, A_4, a battery and a make-and-break key K; while A_1, A_3 are joined up as a secondary circuit to which the former is the primary, and a branch containing a galvanometer is made to join two points P, Q, on this latter circuit.

Ratio of Inductances obtained as Ratio of Two Resistances.

The resistances R_1 R_3 of the coils A_1, A_3, respectively, with any additional resistance included with the coil in each case up to PQ, are adjusted by adding resistance coils from boxes, until there is no current through the galvanometer when the battery circuit is made or broken, and are then compared by means of a Wheatstone's bridge or other convenient method. We have then

$$\frac{M_{34}}{M_{12}} = \frac{R_3}{R_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

To increase the sensibility of this and similar methods, some arrangement such as Ayrton and Perry's secohmmeter, described below, for successively making and breaking the battery circuit, and sending the successive integral flows through the galvanometer in the same direction, must be adopted.

Theory of Method.

To prove the condition (12) let L_1, L_3 be the self-inductances of the coils A_1, A_3, L the self inductance of the battery circuit, and Γ that of the galvanometer. Then if $\dot{u}$ be the battery current at any instant, $\dot{x}$, $\dot{y}$, the currents in the same direction round A_1, A_3, respectively, the current through the galvanometer is $\dot{x} - \dot{y}$, and the electrokinetic energy of the system is given by the equation

$$T = \tfrac{1}{2}\{L\dot{u}^2 + L_1\dot{x}^2 + L_3\dot{y}^2 + \Gamma(\dot{x} - \dot{y})^2 + 2M_{12}\dot{u}\dot{x} + 2M_{34}\dot{u}\dot{y}\} \;. \qquad (13)$$

If R be the resistance of the battery circuit, G the resistance of the galvanometer, we get for the dissipation function

$$F = \tfrac{1}{2}\{R\dot{u}^2 + R_1\dot{x}^2 + R_3\dot{y}^2 + G(\dot{x} - \dot{y})^2\} \;. \quad . \quad . \qquad (14)$$

Since the impressed electromotive forces corresponding to $\dot{x}$, $\dot{y}$, are zero, we have by (4)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} + \frac{\partial F}{\partial \dot{x}} = 0, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{y}} + \frac{\partial F}{\partial \dot{y}} = 0.$$

Equations of Currents.

Hence by (13) and (14)

$$L_1\ddot{x} + \Gamma(\ddot{x}-\ddot{y}) + M_{12}\ddot{u} + R_1\dot{x} + G(\dot{x}-\dot{y}) = 0$$

$$L_2\ddot{y} - \Gamma(\ddot{x}-\ddot{y}) + M_{34}\ddot{u} + R_2\dot{y} - G(\dot{x}-\dot{y}) = 0,$$

or, integrating and writing z for $\dot{x}-\dot{y}$, and taking account of the fact that x, y, $\dot{x}$, $\dot{y}$, $\dot{u}$, are initially zero, we obtain two equations which may be written

$$\left.\begin{aligned} \left\{(L_1+\Gamma)\frac{d}{dt} + R_1 + G\right\} z + \left(L_1\frac{d}{dt} + R_1\right) y + M_{12}\,\dot{u} = 0 \\ -\left(\Gamma\frac{d}{dt} + G\right) z + \left(L_2\frac{d}{dt} + R_2\right) y + M_{34}\,\dot{u} = 0 \end{aligned}\right\} \quad (15)$$

Eliminating y between these we obtain an equation of the form

$$A\ddot{z} + B\dot{z} + Cz = D\ddot{u} + E\dot{u}$$

where A, B, C, D, E are constants.

Deduction of Condition for no Integral Galvanometer-Current.

Soon after completion of the primary circuit the current in the secondary will have died out. Then the last equation becomes

$$Cz = E\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where γ is the steady current in the primary. By inspection of (15) it is easy to see that

$$C = R_2(R_1+G) + R_1 G, \quad E = M_{34}R_1 - M_{12}R_2.$$

Thus (16) becomes, since $z = \dot{x} - \dot{y}$

$$\dot{x}-\dot{y} = \frac{M_{34}R_1 - M_{12}R_2}{R_2(R_1+G) + R_1 G}\gamma \quad . \quad . \quad . \quad . \quad . \quad (16')$$

Hence if $z\,(=\dot{x}-\dot{y}) = 0$,

$$\frac{M_{34}}{M_{12}} = \frac{R_2}{R_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

the relation stated above.

It is to be noticed that if $z = 0$ at each constant, and this relation be fulfilled, $D = 0$, that is by (15)

$$\frac{M_{34}}{M_{12}} = \frac{L_2}{L_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Thus the ratio R_3/R_1 is also the ratio of the self-inductances, if the arrangements be such that no current whatever passes in either direction through the galvanometer.

It is sometimes important, as Lord Rayleigh has pointed out,* that this last condition, and in other cases a similar one if it exist, should be fulfilled in order that the method may be an absolutely null one. Very frequently unless the galvanometer-needle is of very long period it shows considerable uneasiness even if the condition for zero integral current is fulfilled. The fulfilment of (18) or a corresponding condition may be brought about by the insertion of self-inductance in addition to that associated with the conductors employed as resistances.

Comparison of Two Mutual Inductances.

The experiment may be arranged with a derived branch on both the primary and the secondary circuit, as shown in Fig. 110, and the galvanometer in the

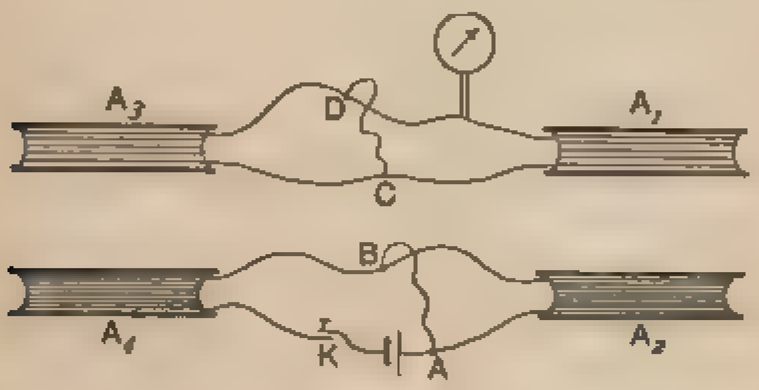


Fig. 110.

Modification of Maxwell Method.

circuit of one of the coils as A_1. Let the resistance of the coil A_2 and connections to the right of AB be R_2, the resistance to the left of AB R_4, the resistances similarly to right and left of CD R_1, R_3 (R_1 including the resistance of the galvanometer), the resistance of the derived branch on the primary R, of the derived branch on the secondary S.

* British Association Report, 1883.

Ratio of Inductances in Terms of Resistances.

With this arrangement when no current through the galvanometer is produced on depressing or raising the battery key the relation

$$-\frac{M_{34}}{M_{12}} = \frac{R(R_3 + S)}{S(R + R_2)} \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

holds. Thus the mutual inductances are opposite in sign, that is the coils must be joined up so that the induced electromotive forces in the secondary circuit are opposed. In the test therefore the coils are joined up in this way, and the resistances are adjusted until no deflection of the galvanometer needle is produced by making or breaking the battery circuit.

Particular Cases.

If $R = \infty$, that is if there is no derived branch on the primary, the relation (19) becomes

$$-\frac{M_{34}}{M_{12}} = \frac{R'_3 + S}{S} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

In this case, since numerically $M_{34} > M_{12}$, the galvanometer must be placed on that side of CD on which the induction is the weaker.

If $S = \infty$, that is if there is no derived branch on the secondary,

$$-\frac{M_{34}}{M_{12}} = \frac{R}{R + R_2} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

If, for the coils in the positions of Fig. 110, $M_{12} > M_{34}$ numerically, (21) becomes $-M_{12}/M_{34} = R/(R + R_4)$.

Theory of Method.

Let i be the current at any instant through the battery, and therefore through A_4, i' the current at the same instant through A_2, then the current in AB is $i - i'$. Denoting now by L_2, L_4, L, the inductances of the three parts into which the primary circuit

is divided, namely A_3, A_4, and the derived branch AB, by L' the inductance of the derived branch CD, we have

$$T = \tfrac{1}{2}\{L_1\dot{x}^2 + L_3\dot{y}^2 + L'(\dot{x}-\dot{y})^2 + L_2\dot{u}'^2 + L_4\dot{u}^2 + L(\dot{u}-\dot{u}')^2$$

$$+ 2M_{12}\dot{u}'\dot{x} + 2M_{34}\dot{u}\dot{y}\} \quad . \quad . \quad . \quad . \qquad (22)$$

$$F = \tfrac{1}{2}\{R_1\dot{x}^2 + R_3\dot{y}^2 + S(\dot{x}-\dot{y})^2 + R_4\dot{u}^2 + R_2\dot{u}'^2 + R(\dot{u}-\dot{u}')^2\}, \qquad (23)$$

where $\dot{x}$, $\dot{y}$ denote as before the currents in A_1, A_3.

The equations of currents obtained from these and integrated over any interval from an instant just before the contact was made or broken, with attention to the fact that the initial values of the variable quantities are all zero, give equations which can be written in the form

$$\left.\begin{aligned} &\left\{(L_1 + L')\frac{d}{dt} + R_1 + S\right\}x - \left(L'\frac{d}{dt} + S\right)y + M_{12}u' = 0 \\ &-\left(L'\frac{d}{dt} + S\right)x + \left\{(L' + L_3)\frac{d}{dt} + R_3 + S\right\}y + M_{34}u = 0 \end{aligned}\right\} \quad . \qquad (24)$$

Elimination of y from these gives an equation of the form

$$A\ddot{x} + B\dot{x} + Cx = D\ddot{u} + D'\ddot{u}' + E\dot{u} + E'\dot{u}'.$$

If the currents have become steady this reduces to

$$Cx = E\gamma + E'\gamma',$$

where x is the time-integral of the current which has passed through the galvanometer, and γ, γ' are the steady currents in the battery and the coil A_2. Hence $\gamma' = \gamma R/(R + R_2)$, and

$$Cx = \left(E + E'\frac{R}{R + R_2}\right)\gamma \quad . \qquad (25)$$

Now by (24)

$$C = (R_1 + S)(R_3 + S) - S^2$$
$$E = -M_{34}S, \quad E' = -M_{12}(R_3 + S).$$

Hence (25) becomes

$$x = -\frac{M_{34}S(R + R_2) + M_{12}R(R_3 + S)}{(R + R_2)\{(R_1 + S)(R_3 + S) - S^2\}}\gamma \quad . \quad . \qquad (25')$$

If $x=0$ this gives at once

Condition for Zero Value

$$-\frac{M_{34}}{M_{12}}=\frac{R(R_3+S)}{S(R+R_4)} \quad . \quad (26)$$

the condition (19) above for no integral current through the galvanometer.

Ayrton and Perry's Secohmmeter.

As stated above, the sensibility of these methods may be greatly increased by using successive reversals of the battery current, with a proper arrangement for commutating the inductive flows through the galvanometer. An excellent contrivance for this purpose has been provided by Professors Ayrton and Perry in the Secohmmeter. This is an arrangement of two rotary commutators, worked by the same spindle, one for periodically interchanging the points to which the galvanometer terminals are attached, the other for reversing the battery circuit. Each of these commutators, as will be seen from the diagrammatic sketches (Fig. 112 showing the mode of using the instrument, consists of four brushes pressing on a cylindrical surface made up of two nearly semicylindrical metal pieces separated by insulating material. The relative times of reversal by the two commutators can be adjusted to suit the purpose for which it is to be used.

The spindle can be driven by a handle or by any convenient small motor. For a given speed of driving, two speeds of the commutators can be arranged for. With one there are rather more than eight, and with the other twenty-four, reversals effected by each for one turn of the handle or driving pulley. The speed of the driving handle or pulley is governed by a fly-wheel.

Use of Secohmmeter for Comparison of Mutual Inductances.

For example, the instrument can be applied to the comparison of two mutual inductances by the methods just described. The battery commutator is arranged to reverse the battery circuit at an instant when the galvanometer circuit on the secondary is complete. An induction-flow takes place through the instrument unless the proper adjustment of resistances has already been made. After the battery current has reached its steady value, the galvanometer terminals are reversed by the commutator preparatory to a second reversal of the battery. The flow due to induction in this second case thus takes place through the instrument in the same direction as before, and so on as the commutator revolves. If the period of rotation is small in comparison with that of oscillation of the needle, the result is to give a steady deflection equal to that which would be produced by a current equal to nq, where n is the number of reversals of the

battery per second, and q the quantity of electricity which passes at each of them.

The sensibility therefore increases with the speed of rotation; but in the present application, as in all others in which only the *integral flow* through the galvanometer, taken over the interval of variation of battery current, vanishes for certain experimental arrangements, the speed must not be so great as to prevent the battery current from reaching its steady value between each pair of reversals. In cases in which the method is really "null" the speed may be made as high as is thought desirable.

M. Brillouin's Experiments.

M. Brillouin* has carried out some careful comparisons of mutual inductances by these methods. He used (1) a derived branch on the primary, (2) a derived branch on the secondary (with in each of these cases the galvanometer in series with one of the coils in the secondary), (3) the galvanometer in the derived branch on the secondary. We give here a short account of experiments (1) and (3).

Arrangement of Apparatus with Derived Branch on Primary.

The derived branch in (1) was made up of a resistance box reading to fractions of an ohm. As its coils were not wound double it was placed at a distance from the rest of the apparatus.

The galvanometer used had a resistance of 900 ohms and was an astatic needle mirror instrument. It was provided with a damping vane of wire gauze, and was enclosed in a case to shield off air currents. The observations were made in the ordinary way by means of a telescope and attached scale placed at a distance of 1 metre from the mirror.

The connecting wires were carried along side by side to reduce their external action as nearly as possible to zero.

Rotating Commutator.

As the galvanometer was not sensitive enough to enable measurements to be made satisfactorily with a single make or break, a rotating commutator driven by a Gramme motor was arranged, so that in each turn it (1) connected the galvanometer with the secondary circuit, (2) closed the primary circuit, (3) short circuited the galvanometer, (4) opened the primary.

The secondary circuit was kept closed permanently and the galvanometer received only the transient current at each closing of the primary. About 10 impulses were given to the needle per second, and a permanent deflection was thus produced.

Description of Coils compared.

The coils used were first a pair consisting of an exterior coil made of a cable of twenty insulated wires lightly twisted together, surrounding an internal bobbin of somewhat thick wire. The mutual inductance between the internal bobbin and each of the twenty strands of the other was the same, M say. A commutator

* *Thèses Présentées à la Faculté des Sciences de Paris*, 1882.

enabled any number p of the strands to be opposed to the rest, so that the coefficient of induction between the two bobbins was reduced to $(20-2p)M$. The wires however being kept in series the resistance did not vary.

The maximum mutual inductance of these coils will be denoted by M_{12}.

In experiments (1) of which results are quoted below a pair of coils was used of mutual inductance intermediate (for the positions adopted) between the maximum and minimum inductances of the apparatus just described. We shall denote the mutual inductance of these coils by M_{34}.

A pair of coils used in experiments (3) consisted of a very carefully wound bobbin of thick wire 19 cms. long, and 10 cms. in internal, 12 cms. in external diameter, placed concentrically with a small coil of length 4·7 cms. and internal and external diameters 1 cm., 5 cms. respectively. The latter bobbin could be turned round through any required angle by means of an index and divided circle. The external coil being long, the two coils had a coefficient of mutual induction proportional to the cosine of the inclination of the axes.

The coefficient of induction between these coils in any given relative positions will be denoted by M'_{34}.

Results of Experiments

The following are the results of five experiments made with different fractions h of M_{12}, and no derived branch on the secondary. The ratio of the coefficients comes out as shown in (21) in terms of the resistance R of the shunt on the primary, and R_2 the resistance of the coil A_2 in Fig. 109. R_2 was corrected to agreement at the temperature of experiment with the box from which R was taken.

h	Temp.	R	$\frac{R+R_2}{R_2}$	$\frac{M_{12}}{M_{34}}$
1	15° C.	68·5 ± 0·1	1·671	1·671
0·9	14·2 „	91·6 ± 0·1	1·500	1·606
0·8	14·7 „	138·9 ± 0·1	1·348	1·672
0·8	14·8 „	139·2 ± 0·1	1·330	1·662
0·8	14·2 „	137·6 ± 0·1	1·333	1·666
				Mean 1·667

A set of experiments was also made with the same arrange-

ment, and at one temperature 12°·6 C. with $hM_{12} < M_{34}$. The ratio in this case comes out in terms of the resistance R_4 of the coil A_4 and any non-inductive resistance in series with it, and the resistance R of the derived branch. R_4 was that of the bobbin A_4, together with a resistance seven times as great, making $R_4 = 18{\cdot}49$ ohms in all.

The results are given in the table.

h	R	$\frac{R}{R+R_4}$	$\frac{M_{12}}{M_{34}}$
0·1	3·69	·1663	1·663
0·2	9·35	·336	1·680
0·2′	9·33	·335	1·675
0·1 + 0·2	18·62	·5017	1·672
0·1 + 0·2′	18·70	·5028	1·676
0·2 + 0·2′	37·5	·669	1·672
0·1 + 0·2 + 0·2′	92·7	·8336	1·667
0·5	92·5	·8334	1·667

These results give by addition for the values 1, ·9, ·8, of h used in the former set of experiments

$$M_{12}/M_{34} = 1{\cdot}670,\ {\cdot}9M_{12}/M_{34} = 1{\cdot}504,\ {\cdot}8M_{12}/M_{34} = 1{\cdot}334$$

which closely agree with the values of $(R+R_2)/R_2$ then found.

A set of experiments was also made with the galvanometer included in a derived branch on the secondary according to the arrangement of which the theory is given above p. 445. Experiments by Maxwell's Method.

The galvanometer was a very sensitive astatic instrument of the Thomson pattern with a coil of 7,000 ohms resistance. The coils, which were the two pairs already described, were at distances of only about 2½ metres from the galvanometer, but were placed in such positions that the direct action of each on the needle was zero. They could be turned through 10° from these positions without producing any sensible action. The induced currents in the small bobbin of the second pair of coils, was found to produce no direct effect upon the needle in any position in which the bobbin was used.

All the joining wires had their outgoing and return parts together and were carefully insulated.

The primary circuit contained a battery of 10 Daniell's cells; and the rotating commutator was not employed, as the galvanometer was sufficiently sensitive to show a single impulse when Method of Experimenting.

the integral current through it was not zero. For the final adjustment the deflections were amplified by closing and opening the circuit when the needle was passing through zero alternately in opposite directions. Any want of perfect adjustment manifested itself by the aggregate effect of the successive exceedingly small impulses thus given, since these all tended to increase the kinetic energy of the needle.

Effect of Currents in Suspended Metal Vane:

But for balance in these circumstances it is necessary that the effects on the needle-system of completing the circuit and of breaking the circuit should both be zero. It was found at first that, while making the circuit produced no effect, breaking it always produced a slight impulse. This M. Brillouin traced to inductive action between the coils and the metallic vane attached to the needles for the purpose of damping. This induction depended on the law of variation of the induced current in the coils and took place notwithstanding the fact that the integral current at break was zero as well as that make. By placing a condenser across the primary circuit and the make and break key, the law of variation of the current could be altered; and it was found that a corresponding change took place in the deflection. The electromagnetic action between the induced currents in the vane and the inducing current in the coils clearly ought to cause such effects as those observed.

Elimination of Disturbances.

It was found that this action had a maximum for any position of the needles when the capacity of the condenser was ·25 microfarad, and that when the vane was quite symmetrically placed relatively to the coils the effect always vanished. A condenser of this capacity was therefore applied, and the position of the needles adjusted by the directing magnet until the effect was zero. The experiment was then made, and the method of multiplication used for the deflections, with certainty that the effect of make was exactly equal and opposite to that of break.

By (12) above we have

$$\frac{hM_{12}}{M'_{34}} = \frac{R_1}{R_3}.$$

Results of Experiments.

In the experiments made R_3 was constant and $=974{\cdot}2$ ohms, while R_1 was made up of a constant part $R=1264{\cdot}1$ ohms, and a variable part r. The results of one set of experiments are given in the table. The fourth column is calculated by taking the fraction h of the sum of the results in column 3, and indicates the closeness of agreement of the results.

h	r	$\frac{hM_{12}}{M'_{34}}$	$\frac{hM_{12}}{M'_{34}}$ (Mean value from last col.).
0·1	191 ± 0·5	1·493 ± ·001	1·491
0·2	1639 ± 3	2·980 ± ·003	2·981
0·2'	1640 ± 3	2·981 ± ·003	2·981
0·5	5994 ± 3	7·450 ± ·003	7·452

The following method of comparing two self-inductances is due to Clerk Maxwell.* The two coils the inductances, L_1, L_2, of which are to be compared are placed in adjacent branches, AC, AD, of a Wheatstone bridge (Fig. 111), and balance is obtained for steady

Comparison of Two Self-Inductances.

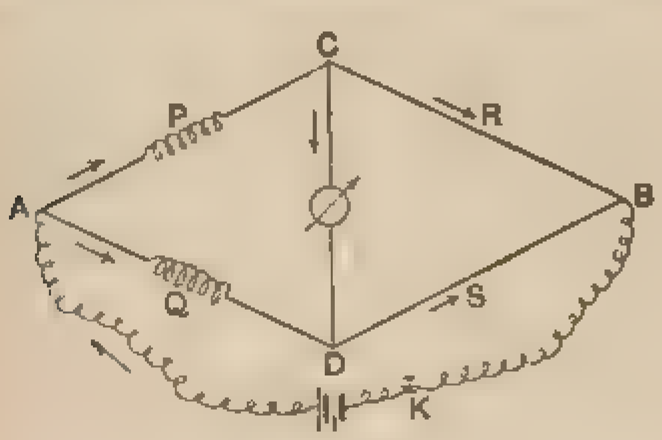


FIG. 111.

currents by properly adjusting the (non-inductive) resistances R, S of the branches CB, DB. If the resistances of the branches AC, AD be P, Q respectively,

* *El. and Mag.* Vol. ii. p. 367 (Second Edition.)

and the inductances of the branches CB, DB are negligible, the relation fulfilled when the balance is attained is, as we know, $PS = QR$. If besides this the relation

$$\frac{L_1}{L_2} = \frac{R}{S} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

be fulfilled, there will be also balance for transient currents, and no deflection of the needle will be produced when, the galvanometer branch CD being complete, the battery circuit is made or broken. Or the coils may be placed in AC, CB so that L_1 is associated with P and L_2 with R; then balance is obtained when

$$\frac{L_1}{L_2} = \frac{P}{R} \quad . \quad . \quad . \quad . \quad . \quad (27')$$

A secohmmeter may be used, as shown in Fig. 112 to increase the sensibility. Balance for induction currents is simply tested for by rotating the commutators. The arrangement of the apparatus will be obvious from the diagram.

Theory of Method

To prove (27) and (27′) we write down by (6) the equations of currents of the circuits $ACDA$, $CBDC$ putting Γ, G, for the self-inductance and resistance of the galvanometer, L_3, L_4, for the inductances of the branches CB, DB, $\dot{x}$ for the current in AC, $\dot{y}$ for that from C to D, and $\dot{u}$ for that in the battery at any instant. The equations are by (6)

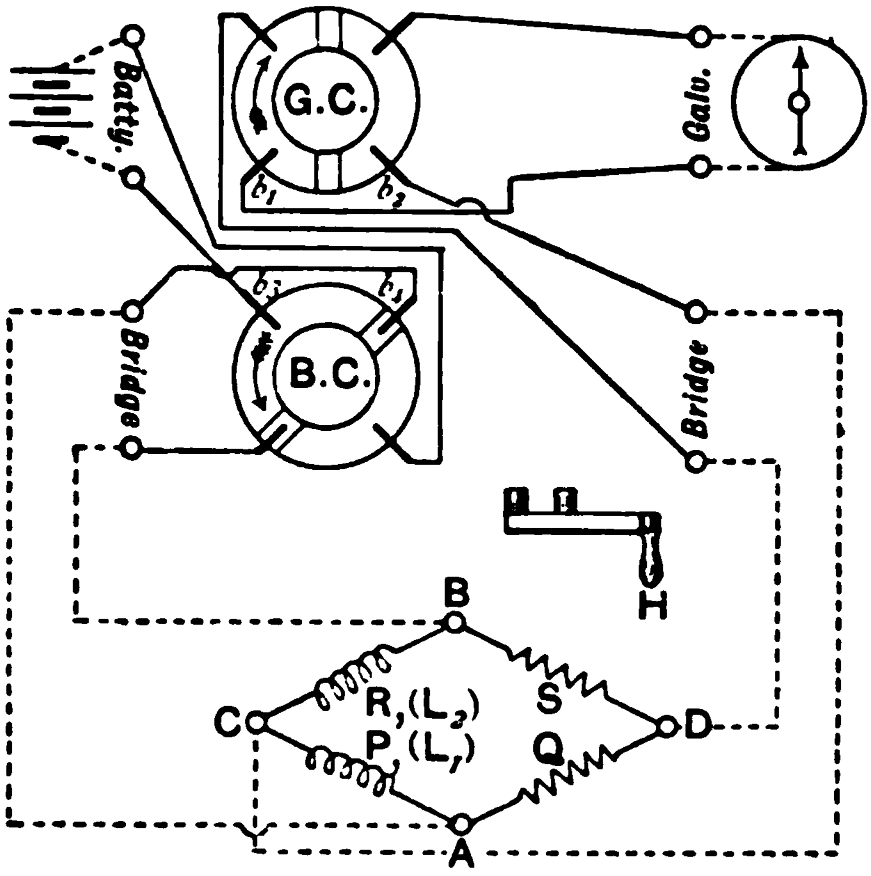
Equations of Currents.

$$\left.\begin{array}{l} L_1\ddot{x} + P\dot{x} + \Gamma\ddot{y} + G\dot{y} - L_2(\ddot{u} - \ddot{x}) - Q(\dot{u} - \dot{x}) = 0 \\ L_3(\ddot{x} - \ddot{y}) + R(\dot{x} - \dot{y}) - L_4(\ddot{u} - \ddot{x} + \ddot{y}) - S(\dot{u} - \dot{x} + \dot{y}) - \Gamma\ddot{y} - G\dot{y} = 0 \end{array}\right\} \quad (28)$$

Integrated over the whole period of variation of currents these equations become, since there is finally zero current in CD,

$$\left.\begin{aligned}(P+Q)x \qquad\qquad &+ Gy = Qu + \frac{PL_2 - QL_1}{P+Q}\gamma \\ (R+S)x - (G+R+S)y &= Su + \frac{PL_4 - QL_3}{P+Q}\gamma\end{aligned}\right\} \quad . \quad . \qquad (29)$$

where x, y, u denote the quantities of electricity which have flowed through AC, CD, and the battery, respectively, in the interval of integration, γ denotes the steady current through the battery, and for the steady current in the branch AC has been put its value $\gamma Q/(P+Q)$.



The continuous lines represent permanent connections inside instrument, the dotted lines temporary connections, bridge, &c.

FIG. 112.

Elimination of x from (29) gives

$$y = -\gamma \frac{L_1 - P\left(\dfrac{L_2}{Q} + \dfrac{L_3}{R} - \dfrac{L_4}{S}\right)}{\dfrac{1}{S}\{G(P+Q+R+S) + (P+Q)(R+S)\}} \quad . \qquad (30)$$

Hence in order that y may be zero we must have

$$L_1 - P\left(\frac{L_2}{Q} + \frac{L_3}{R} - \frac{L_4}{S}\right) = 0 \quad . \quad . \quad . \quad (31)$$

or if L_3, L_4 be negligible

$$\frac{L_1}{L_2} = \frac{P}{Q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31')$$

the relation stated above.

Condition for Zero Integral Flow through Galvanometer.

It is to be noticed that if L_1 be negligible in comparison with the other inductances, and P be finite, balance will be obtained if the resistances Q, R, S are such that

$$\frac{L_2}{Q} + \frac{L_3}{R} - \frac{L_4}{S} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (31'')$$

This result will be of use in connection with the comparison of a mutual inductance with a self-inductance.

Sensibility of Arrangement.

We may now shortly investigate the sensibility of the arrangement. If r be the resistance of the battery branch AB, the resistance of the whole circuit for steady currents is evidently $r + PQ/(P+Q) + RS/(R+S)$, or since $PS = QR$, $r + S(P+R)/(R+S)$. If E be the electromotive force of the battery

$$\gamma = E/\{r + S(P+R)/(R+S)\}.$$

Thus (30) becomes with a little reduction

$$y = -E\frac{L_1 - P\left(\frac{L_2}{Q} + \frac{L_3}{R} - \frac{L_4}{S}\right)}{\left\{P + R + r\left(1 + \frac{R}{S}\right)\right\}\left\{G\left(1 + \frac{P}{R}\right) + P\left(1 + \frac{S}{R}\right)\right\}} \quad (32)$$

If the ratio $R/S(=P/Q)$ be taken as fixed, and P and G given, R is to be taken so that the denominator, D say, of this expression for y may be a minimum. Denoting R/S by ρ, we have

$$D = \left\{G\left(1 + \frac{P}{R}\right) + P\left(1 + \frac{1}{\rho}\right)\right\}\{r(\rho+1) + P + R\}$$

Calculating dD/dR from this and equating it to zero we find

$$R^2 = \frac{GP\rho\{P + r(\rho+1)\}}{G\rho + P(\rho+1)} \quad . \quad . \quad . \quad . \quad (33)$$

Conditions that Galvanometer Current may be always Zero.

If the condition (31), and the relation $L_2L_3 - L_1L_4 = 0$, are fulfilled, the difference of potential between C and D is always zero and therefore not only is there no integral flow from C to D, but the current at each instant is zero. This may be seen as follows. Assuming that the difference of potential at any instant is zero, there will be no current through the galvanometer. Hence

$$L_1\dot{x} + Px = L_2(\dot{u} - \dot{x}) + Q(u - x),$$

and

$$L_3\dot{x} + Rx = L_4(\dot{u} - \dot{x}) + S(u - x).$$

Eliminating u from these equations we get the relation

$$(L_3L_2 - L_1L_4)\frac{d^2x}{dt^2} + (L_3Q + L_2R - L_1S - L_4P)\frac{dx}{dt} + (QR - PS)x = 0$$

which must hold for all values of x, dx/dt, d^2x/dt^2. Hence we must have in the first place,

$$QR - PS = 0$$

the condition for balance in the case of steady currents.

Equating the coefficient of dx/dt to zero, and using the relation $QR = PS$ we get

$$\frac{L_1}{P} - \frac{L_2}{Q} - \frac{L_3}{R} + \frac{L_4}{S} = 0,$$

which is the condition [(31)] that there should be no integral flow through the galvanometer at make (or break) of the battery circuit.

Lastly, equating the coefficient of d^2x/dt^2 to zero, we find

$$L_2L_3 - L_1L_4 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Use of Telephone in Wheatstone's Bridge.

which shows that if C and D are kept at one potential always, the inductances of the branches of the bridge must fulfil a relation precisely similar to that fulfilled by the resistances when there is balance for steady currents. The relations (31) and (34) must be fulfilled by the inductances in order that a telephone may be used in a Wheatstone's bridge. When the telephone was first introduced it was thought by many experimenters that by using a telephone and intermittent currents the Wheatstone's bridge method of testing could be made much more sensitive. As a matter of fact there can be silence in a telephone substi-

tuted for a galvanometer in a Wheatstone's bridge, only if the inductances are balanced as well as the resistances by being made to fulfil the relation (34).

If L_3, L_4, are negligibly small each term of (34) vanishes, and the only condition to be fulfilled by the inductances is then (31) which takes the form

$$\frac{L_1}{L_2} = \frac{P}{Q}.$$

The converse proposition however that if this condition, or in the more general case (31) and (34), be fulfilled, the current through the galvanometer is always zero is not proved. But if the points C, D, are not joined by a wire, and the conditions be fulfilled, CD will, it has just been shown, be at the same potential during the whole interval of variation of the currents. Hence, if at any instant during that interval a conductor, of any resistance and inductance, be supposed applied between C and D, no current would start in it, since there would be no difference of potential between its extremities. Thus, with fulfilment of the condition, varying flow in the network, with zero current in CD, is physically possible, and is the solution of the problem, otherwise there would be more than one solution, and this we know to be impossible if the currents can be regarded as a dynamical system.

Practice of the Method.

In the practice of the method the battery key is depressed first, then the galvanometer key, and balance is obtained in the ordinary way for steady currents. Then a test of balance is made for variable currents by putting down the galvanometer key first and observing whether there is any sudden deflection to one side or the other when the battery key is depressed.

If there is, the resistances R, S are unaltered, and balance for steady currents restored by adding non-inductive resistance to the coils in AC, AD. Then a test is made for an induction deflection as before, and if necessary a further change in R, S is made, and so on. Balance for steady currents is, at each step of the adjustment, obtained before a test for the variable cur-

tents is made, and thus confusion between a transient and a steady deflection is avoided.

The repeated adjustments necessary in this method render it troublesome in the above form. The following modification of it, due to Prof. C. Niven,* overcomes this difficulty. One of the coils say that of inductance L and resistance P is made one arm of a Wheatstone bridge (Fig. 113), and balance is obtained with resist- Niven's Modification of Maxwell's Method.

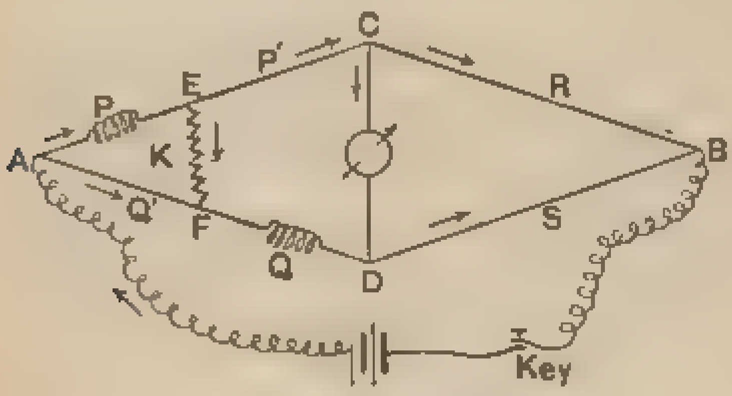


FIG. 113.

ances Q', R, S which form the other three branches. The other coil of inductance L' and resistance Q is then inserted at FD, and balance is restored by inserting a non-inductive resistance P' at EC. Non-inductive resistance K, is then inserted between E and F until there is no induction current in the galvanometer, when putting down the battery key produces no current through the previously completed circuit of the galvanometer. When this is the case

$$\frac{L}{L'} = \frac{(K+P+Q)R}{KS} \quad . \quad . \quad (35)$$

* *Phil. Mag.* Sept. 1887.

Theory of the Method. Let at any instant $\dot{u}$ be the current through the battery, $\dot{x}$ the current from A to E, $\dot{y}$ that from C to D, $\dot{z}$ that from E to F, then the other currents are, in EC $\dot{x}-\dot{z}$, in FD $\dot{u}-\dot{x}+\dot{z}$, in CB $\dot{x}-\dot{y}-\dot{z}$, in DB $\dot{u}-\dot{x}+\dot{y}+\dot{z}$. We get then by (6) from the three circuits $AEFA$, $ECDFE$, $CBDC$, the following equations of currents in which Γ, G, denote respectively the inductance and resistance of the branch CD.

$$L\dot{x} + P\dot{x} + K\dot{z} - Q'(\dot{u}-\dot{x}) = 0$$

$$P'(\dot{x}-\dot{z}) + \Gamma\ddot{y} + G\dot{y} - L'(\dot{u}-\dot{x}+\dot{z}) - Q(\dot{u}-\dot{x}+\dot{z}) - K\dot{z} = 0$$

$$R(\dot{x}-\dot{y}-\dot{z}) - S(\dot{u}-\dot{x}+\dot{y}+\dot{z}) - \Gamma\ddot{y} - G\dot{y} = 0.$$

Integrating these from an instant just before closing the circuit of the battery to the steady state, denoting the steady currents in AE and the battery by i_s and γ respectively, and remembering that the adjustments have been supposed so made that the steady currents in EF, CD, are zero, we get

$$\left.\begin{aligned}
&(P+Q')x + Kz = Q'u - Li_s\\
&Gy + (P'+Q)x - (Q+P'+K)z = Qu + L'(\gamma - i_s)\\
&-(G+R+S)y + (R+S)x - (R+S)z = Su
\end{aligned}\right\} \quad . \quad (36)$$

Integral Equations of Currents. Hence if Δ denote the determinant of this system of equations, we get by elimination of x and y

$$\Delta y = \begin{vmatrix} Q'u - Li_s, & P+Q', & K \\ Qu + L'(\gamma - i_s), & P'+Q, & -(P'+Q+K) \\ Su, & R+S, & -(R+S) \end{vmatrix}$$

Expression for Integral Current in Galvanometer. Expanding this determinant (first simplifying it by adding the second column to the third), remembering that since $PQ' = P'Q = R'S$, the relations $(R+S)Q' = (P+Q')S$, $(P'+Q)S = (R+S)Q$, hold, and putting $(\gamma - i_s)/i_s = R/S$, $i_s = \gamma S/(R+S)$ we find

$$\Delta y = \gamma\{(K+P+Q')RL' - KSL\} \quad . \quad . \quad . \quad . \quad (37)$$

If the right hand side be zero, and, as will generally be the case, the determinant Δ does not vanish, y must be zero. Hence in order that there may be no integral current through the galvanometer, it is necessary and sufficient that as stated in (35)

$$\frac{L}{L'} = \frac{(K+P+Q')R}{KS}$$

Arrangement of Bridge for Sensibility.

If r denote the resistance of the battery branch AB, we easily see, taking account of the relations $PQ' = P'Q = RS$, that the resistance of the whole circuit for steady currents is

$$r + R(Q + Q' + S)/(R + S),$$

and that

$$\Delta = \frac{R+S}{S}(K + P + Q)\left\{GS + \left(Q + \frac{KQ'}{K+P+Q'}\right)(G+R+S)\right\}.$$

Hence putting E for the electromotive force of the battery we have $\gamma = E/\{r + R(K + Q + Q')/(R + S)\}$, and instead of (36)

$$\gamma = E\frac{RL' - \frac{KS}{K+P+Q}L}{\frac{1}{S}\{r(R+S)+R(Q+Q'+S)\}\{GS+W(G+R+S)\}} \quad . \quad (37')$$

in which W is written for $Q + KQ'/(K + P + Q')$.

If D denote the denominator in this expression, then in order that the arrangement may be as sensitive as possible D/R must be made a minimum. For simplicity let $P = Q'$, $P' = Q$, $R = S$. Then S is to be so chosen that D/S shall be a minimum. This by the ordinary method is found to be the case when

$$S^2 = \frac{(2r + Q + Q')\,GW}{G + 2W} \quad . \quad . \quad . \quad . \quad (38)$$

Comparison of Two Inductances by Differential Galvanometer.

This comparison may also be effected by means of a differential galvanometer. The two coils of inductances L_1, L_2 and resistances R_1, R_2, are joined as shown in the diagram with non-inductive adjustable resistances, and balance is obtained for steady currents without the cross-conductor of resistance S. It is plain that if, as we suppose, the resistance of each coil of the galvanometer is the same (G), and their effects on the needle are equal for equal currents, the additional resistances R'_1, R'_2 (including connections) must be equal to R_2, R_1 respectively. If E be the electromotive force and r the

resistance of the battery the steady current in each coil is

$$\gamma = \frac{E}{R_1 + R'_1 + G + 2r} = \frac{E}{R_2 + R'_2 + G + 2r}$$

$$= \frac{E}{R_1 + R_2 + G + 2r} \quad . \quad . \quad . \quad (39)$$

The cross conductor is then applied at the points of junction P, Q, and the balance for steady currents is again tested and if found to be disturbed is restored by slightly

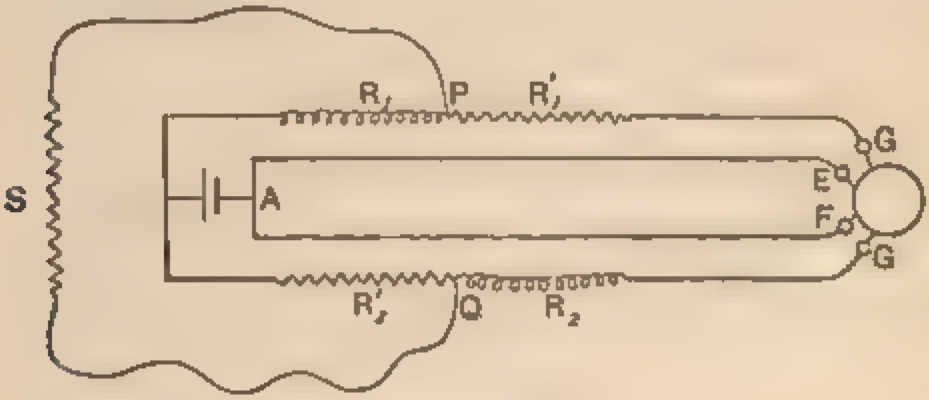


FIG. 114.

shifting one or both of the contacts P, Q. The resistance S is then adjusted until there is no deflection of the needle on depression of the battery key. When this adjustment has been made the relation is fulfilled

$$\frac{L_1}{L_2} = \frac{2R_1 + S}{S} \quad . \quad . \quad (40)$$

Theory of Method. If x, y, be the currents from P, Q, respectively, to the galvanometer, z that from P to Q through the cross-connection, the current arriving from the battery is $x + z$ at P, and $y - z$ at Q. Hence if Γ be the inductance of each galvanometer coil, M their mutual inductance, and the inductances of other parts of the circuits be negligible, the equations of currents for the circuits $APGEA$, $AQGFA$, $APQA$ are by (6)

$$\Gamma \dot{x} + M\dot{y} + L_1(\dot{x} + \dot{z}) + (R_1 + R'_1 + G)\dot{x} + r(\dot{x} + \dot{y}) + R_1\dot{z} = E$$

$$\Gamma \dot{y} + M\dot{x} + L_2\dot{y} + (R_2 + R'_2 + G)\dot{y} + r(\dot{x} + \dot{y}) - R'_2\dot{z} = E$$

$$L_1(\dot{x} + \dot{z}) + S\dot{z} + R_1(\dot{x} + \dot{z}) - R'_2(\dot{y} - \dot{z}) = 0.$$

Equations of Currents.

Integrating the first two of these equations over the rise of the current in each circuit from zero to the steady value γ, and subtracting the second integral from the first we get since $R'_1 = R_2$, $R'_2 = R_1$

$$(L_1 - L_2)\gamma + (R_1 + R_2 + G)(x - y) + (R_1 + R'_2)z = 0 \qquad (41)$$

Also the third equation integrated gives

$$L_1\gamma + R_1(x - y) + (2R_1 + S)z = 0 \quad . \quad . \quad . \quad (42)$$

Substituting in (41) the value of z given by (42) and solving for $x - y$ we obtain

Effective Current in Differential Galvanometer.

$$x - y = \frac{(2R_1 + S)L_2 - SL_1}{2R_1(R_2 + G) + S(R_1 + R_2 + G)}\gamma \quad . \quad . \quad (43)$$

In order that this may be zero we must have

$$\frac{L_1}{L_2} = \frac{2R_1 + S}{S} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

The value $E/(R_1 + R_2 + G + 2r)$ substituted for γ in (43) gives

$$x - y = E\frac{(2R_1 + S)L_2 - SL_1}{\{2R_1(R_2 + G) + S(R_1 + R_2 + G)\}(R_1 + R_2 + G + 2r)} \quad (45)$$

The resistances R_1, R_2 are fixed, and in practice G also is given. If the galvanometer is too sensitive the magnetic field at the needles may be increased in intensity, or the coils may be shunted provided the shunt is precisely the same in inductance (if any) and resistance in both cases. The flow through each coil will, if S' be the resistance of the shunt, be simply $(x - y)S'/(G + S)$, as it would be if the galvanometer coils and shunt had no inductance.

Maxwell has also given the following method of comparing the mutual inductance M of two coils with the self-inductance of one of them. One of these coils, C_1,

Comparison of Mutual Inductance of Two coils with Self-Inductance of one of them.

of inductance L_1 ($> M$) is included in the branch AC (Fig. 115) of a Wheatstone bridge, and the other coil, C_2, of the pair is joined up with the battery in the branch AB. The galvanometer is in the branch CD. Let P, Q, R, S be the resistances of the branches AC, AD, CB, DB, and let balance be obtained for steady currents so that $PS = QR$. Then if the coils be properly

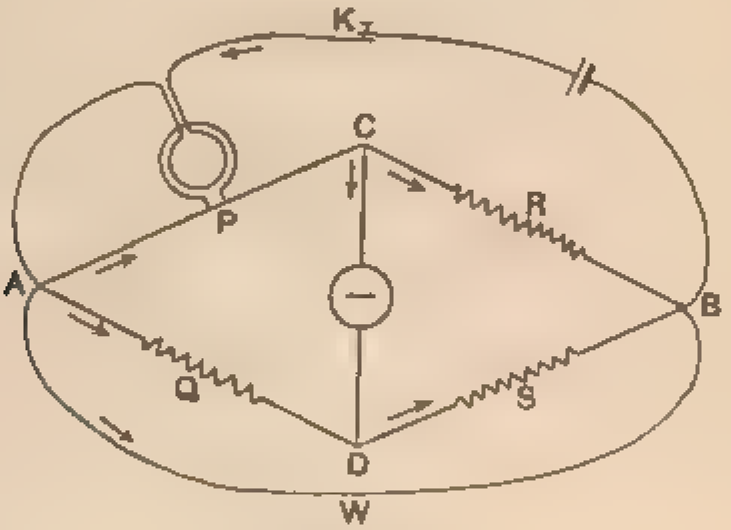


FIG. 115.

placed the ratio $P/Q = R/S$ can be so adjusted that there is no varying current through the galvanometer, and the relation

$$L_1 = -M\left(1 + \frac{P}{Q}\right) = -M\left(1 + \frac{R}{S}\right) \quad . \quad (46)$$

is fulfilled if the inductances of the other branches are negligible, or are balanced in the manner described below.

In order that the bridge may be balanced for both steady and varying currents, the coils must be so placed

Avoidance of Successive Adjustments by Shunting Coil.

that the inductive actions in the branch AC are opposed, and the resistances adjusted until no deflection is produced on depressing or raising the battery key. After each alteration of the ratio P/Q or R/S balance for steady currents must be restored before testing for varying currents. To avoid the repeated adjustments necessary in this process, a non-inductive coil is joined between A and B, and varied in resistance until no deflection is obtained on depressing or raising the battery key after the galvanometer circuit has been completed. The presence of this coil does not affect the balance for steady currents, so that when PS has once been made equal to QR, this adjustment is not disturbed. Now if W be the resistance supplied by this coil and E the point in it at the potential of C, D, it is divided into two parts AE, EB by the point E the resistances of which are $QW/(Q+S)$, $SW/(Q+S)$. Since if we please E may be taken as in contact with D the former of these may be regarded as a shunt on AD, bringing it down to the resistance $QW/(Q+S+W)$, which gives by (46) the relation

$$L = -M\left(1 + \frac{P}{Q} + \frac{P+R}{W}\right). \quad . \quad . \quad (47)$$

Brillouin's Modification of Method.

It will be noticed that there is want of generality of application in this method, inasmuch as both (46) and (47) require that $L > M$. It has been pointed out by M. Brillouin that the method is made perfectly general, and the relation between L and M simplified by putting the coil C_2 in the shunt branch between A and B. Balance for steady currents is first obtained, and then

the total resistance W of the shunt branch is altered until balance is also obtained on making or breaking the battery circuit. The relation between L and M is then

$$L = -\frac{P+R}{W} M \quad . \quad . \quad . \quad . \qquad (48)$$

It is of great importance in this method that the inductances of the other branches of the bridge should be as nearly as possible zero, as sensible inductance of unknown amount unallowed for may very seriously affect the accuracy of the result obtained. The coils used for balance should therefore be as nearly as possible non-inductive.

Elimination of Unknown Inductances of Bridge: It is shown below that if the branches AD, CB, DB have inductances L_2, L_3, L_4, the complete condition for balance when the battery key is depressed or raised, is

$$kM + L_1 - P\left(\frac{L_2}{Q} + \frac{L_3}{R} - \frac{L_4}{S}\right) = 0. \quad . \qquad (49)$$

where k denotes the factor $1 + P/Q + (P+R)/W$, or simply $(P+R)/W$, according as the coil C_2 is placed in the battery circuit or in its shunt AEB. Now we may begin by arranging so that L_2, L_3, L_4 shall be large in comparison with L_1. This may be done by arranging a finite and as nearly as possible non-inductive resistance P in AC greater than that of the coil C_1, while inductive coils are included in the other three branches. Balance for steady as well as for varying currents is then obtained for this arrangement, and we know that then by (31″)

$$\frac{L_2}{Q}+\frac{L_3}{R}-\frac{L_4}{S}=0 \quad . \quad . \quad . \quad . \quad . \quad (50)$$

This operation without some special appliance will involve successive adjustments to balance for steady currents at every alteration of the resistances, but this may be avoided by using for one of the coils, say that in DB, a coil of variable inductance such as two coils joined in series, one of which is within the other and capable of being turned round to any angle of inclination of the axes. The self-inductance of such a pair of coils is made up of two parts, the sum of the self-inductances of the component parts, and twice the mutual inductance between them. The latter part can be varied by varying the positions of the coils; and by this means when once balance for steady currents has been obtained, that for varying currents may be obtained also without altering the resistances of the branches.

This done, C_1 may be included in AC (thus making L_1 finite) and balance for steady currents restored by adjusting P to its former value. Balance for transient currents is then made by varying W, and we have accurately $L = kM$, since L_2, L_3, L_4, Q, R, S have not been altered.

A different method of correction was employed by M. Brillouin. If the coils of a resistance box made of wire doubled in itself before being wound have identical dimensions and be made of wire of the same specific conductivity, but differ only in length and diameter of wire, and moreover be, as of course they generally are, without mutual inductance of sensible amount, the ratio of the small residual inductance of any coils which may be used from the box to their resistance will be approximately the same. This was found to be the case for a resistance box used by Brillouin in his investigations, and Brillouin's Method.

accordingly this box was used to give L_3, R. Balance both for steady and varying currents having first been obtained with certain values of P, Q, R, S, W, and L_1, L_2, L_3, L_4, a resistance r of inappreciable inductance was added to P, and the balances restored by varying R and W to new values R' and W'. The equations were then

$$\frac{kM + L_1}{P} - \frac{L_2}{Q} - \frac{L_3}{R} + \frac{L_4}{S} = 0$$

$$\frac{k'M + L_1}{P + r} - \frac{L_2}{Q} - \frac{L'_3}{R'} + \frac{L_4}{S} = 0$$

which since $L_3/R = L'_3/R'$ gave

$$L_1 = \left\{ (k' - k) \frac{P}{r} - k \right\} M \quad . \quad . \qquad (51)$$

A general investigation given by M. Brillouin shows that in order that this comparison may be carried out with all the exactness of which the method is capable, the galvanometer ought, if used without a commutator giving a steady deflection, to be from 100 times to 1000 times as sensitive for transient as for steady currents. Thus to obtain a sufficiently great galvanometer deflection, a rapidly rotating commutating arrangement, such as Ayrton and Perry's Secohmmeter (p. 457 above), must be employed, if very high accuracy is aimed at.

Theory of Method. Referring to Fig. 115 let the inductances of AC, AD, CB, DB, AEB, and the galvanometer branch CD, be denoted by L_1, L_2, L_3, L_4, L_5, Γ, respectively, and let $\dot{\gamma}$, $\dot{x}$, $\dot{y}$, $\dot{z}$, be the currents in the battery, AC, CD, and the shunt branch AEB at any instant, then integrating over the whole interval of variation of currents at "make" of the battery circuit, and putting γ, x_0, y_0, z_0 for the corresponding values of the steady currents, we get for the integral equations of currents for the circuits $ACDA$, $CBDC$, $ACBEA$.

$$\left.\begin{aligned}(P+Q)x \qquad + Gy + Qz &= -M\gamma + L_2(\gamma - i_2) \\ &\quad - (L_1 + L_2)i_1 + Qu \\ (R+S)x - (G+R+S)y + Sz &= L_4(\gamma - i_2) - (L_3 + L_4)i_1 + Su \\ (P+R)x \qquad - Ry - Wz &= L_5 i_2 - M\gamma - (L_1 + L_3)i_1\end{aligned}\right\} \quad (52)$$

But since the resistance of the bridge network is

$$S(P+R)/(R+S), \quad (\gamma - i_2)/i_2 = W(R+S)/S(P+R),$$

and therefore

$$i_2 = \frac{S(P+R)}{S(P+R) + W(R+S)}\gamma.$$

Again $(\gamma - i_2 - i_1)/i_1 = P/Q$ which gives

$$i_1 = \frac{Q}{P+Q}\,\frac{W(R+S)}{S(P+R) + W(R+S)}\gamma.$$

Substituting these values of i_1, i_2, in (52) and eliminating x and z, we see that since $PS = QR$, the coefficient of u identically vanishes, and we find after easy reductions

$$y = \frac{QW(R+S)}{P+Q}\,\frac{L_1 + M\left(1 + \frac{P}{Q} + \frac{P+R}{W}\right) - P\left(\frac{L_2}{Q} + \frac{L_3}{R} - \frac{L_4}{S}\right)}{\Delta}\gamma \quad (53)$$

where Δ denotes the determinant

$$\begin{vmatrix} G, & P+Q, & Q \\ -(G+R+S), & R+S, & S \\ -R, & P+R, & -W \end{vmatrix}$$

If the coil C_2 is included in the shunt branch AEB, the term involving M in the first and third equation of (52) is $-Mi_2$ instead of $M\gamma$. Hence in the value of y given by (53) we have only to multiply M by i_2/γ to find the proper relation for this case. But

Modification of Formula for Brillouin's Arrangement.

$$i_2/\gamma = S(P+R)/\{S(P+R) + W(R+S)\}.$$

The multiplier of M in the numerator of the second fraction on the right of (53) therefore becomes

$$\left(1+\frac{P}{Q}+\frac{P+R}{W}\right)\frac{S(P+R)}{S(P+R)+W(R+S)}$$

$$=\frac{W(P+Q)+Q(P+R)}{W(R+S)+S(P+R)}\frac{S}{Q}\frac{P+R}{W}=\frac{Q}{S}\frac{S}{Q}\frac{P+R}{W}=\frac{P+R}{W}$$

since $P/R=Q/S$.

In order that γ may vanish the necessary and sufficient condition is thus

$$L_1+M\left(1+\frac{P}{Q}+\frac{P+R}{W}\right)-P\left(\frac{L_2}{Q}+\frac{L_3}{R}-\frac{L_4}{S}\right)=0 \quad . \qquad (54)$$

in the case of Maxwell's arrangement, or

$$L_1+M\frac{P+R}{W}-P\left(\frac{L_2}{Q}+\frac{L_3}{R}-\frac{L_4}{S}\right)=0 \quad . \quad . \quad . \qquad (54')$$

in Brillouin's modification.

It is therefore necessary in order that no error of serious magnitude may enter into the results that L_2, L_3, L_4 may be either negligible or capable of approximate estimation. If the latter is the case the correcting term can be at once found from (54) or (54′).

Most Sensitive Arrangement of Bridge.

We may investigate the most sensitive arrangement of the bridge for this comparison. This we shall do by (a) finding the value of R for a given value, ρ, of the ratio P/Q, and a given galvanometer, (b) finding the proper resistance of a galvanometer bobbin of given shape and dimensions for use with the bridge. Let r be the resistance of the battery (and the coil if included between A and B), then, if W be supposed infinite for the present, the resistance of the circuit is $r+S(P+R)/(R+S)$. Hence if E be the electromotive force of the battery we have $\gamma=E(R+S)/\{r(R+S)+S(P+R)\}$. Hence (53) becomes

$$\gamma=\frac{L_1+kM-P\left(\frac{L_2}{Q}+\frac{L_3}{R}-\frac{L_4}{S}\right)}{\{r(\rho+1)+P+R\}\left\{G\left(\frac{P}{R}+1\right)+P\frac{\rho+1}{\rho}\right\}}E \qquad (55)$$

The condition that the denominator of this expression may be a minimum is easily found in the ordinary way, and is

$$R^2 = \frac{GP\{r(\rho+1)+P\}\rho}{G\rho+P(\rho+1)}. \quad . \quad . \quad . \quad . \quad (56)$$

(1) With given Galvanometer.

(2) With special Galvanometer.

This gives the best value of R for use with a given galvanometer. If however there is a choice of galvanometer-bobbins of the same volume and arrangement of wire but of different resistance, G, then for a given current the galvanometer effect produced by each bobbin varies as $\sqrt{G}$, provided the thickness of the insulating coating be in a constant ratio to the diameter of the wire, or be so small as to be negligible. Thus in order to find the condition for a maximum we have to substitute for the denominator (D say) of the expression on the right of (55) a new denominator $D' = D/\sqrt{G}$. Thus, calculating dD'/dG and equating to zero, we find in addition to (56) the condition

$$G = \frac{\rho+1}{\rho}\,\frac{PR}{P+R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

These give for R the quadratic

$$2R^2 + PR - P\{r(\rho+1)+P\} = 0. \quad . \quad . \quad . \quad (58)$$

This has two real roots, one positive, the other negative. The former is therefore the required value of R and substituted in (57) gives the value of G.

A good practical example is that in which one of the two coils has a comparatively small resistance, as for example the primary of a Ruhmkorff induction coil. If this be put in the battery circuit, and the cells have a low internal resistance, r may be put equal to zero, and we have then

$$\left.\begin{aligned} R &= \frac{1}{2}P \\ G &= \frac{1}{3}\,\frac{\rho+1}{\rho}\,P \end{aligned}\right\} \quad . \quad . \quad (59)$$

These results are not affected in the least by the introduction of the wire of resistance W, since we should then have instead of Q, S, simply, $QW/(Q+S+W)$, $SW/(Q+S+W)$ respectively, and ρ would have the value

$$P(Q+S+W)/QW = R(Q+S+W)/SW,$$

so that (56) and (57) would not be altered.

Practical Example of Method

The following are samples of results obtained by M. Brillouin in experiments made with *two* coaxial and concentric coils of the following dimensions:—

	Mean Diameter.	*Length*	*No. of Turns.*	
Large bobbin......	10·9 cms.	48·5	3263	in four layers
Small „	4·98 „	48·5	3272	in each case.

Value of M calculated (without allowing for thickness of layers) 4·79 × 10 C.G.S.

In all the experiments here quoted the coil C_2 was placed in the battery circuit as shown in Fig. 115.

Q	R	S	W'	k
117·72	100	100	81·886	4·659 ± ·002
117·72	1000	1000	420 ± ·3	4·661 ± ·002
117·73	10000	10000	3806 ± 10	4·658 ± ·005
2·35	100	200	68·85 ± ·05	4·661 ± ·002
587·4	200	1000	91·79 ± ·04	4·659 ± ·002

A series of eight experiments from which these results are selected gave a mean value of $k = 4·6595$.

Four other experiments made with R and S, 1000 ohms and 10000 ohms respectively, and with values of Q, 1176·3, 1176·3, 1165, 1164·8 ohms, gave results agreeing very well with one another, but furnishing a somewhat different mean value of k, namely 4·6397.

Two experiments in which these mean values of k were respectively used to find $L_1 M$, gave

Q	R	S	W''	k'	$L_1 M$
379·6	1000	1000	516·2 ± ·5	4·602 ± ·004	4·661 ± ·003
3785	1000	10000	394·5 ± ·5	4·594 ± ·004	4·660 ± ·005

A rapidly rotating commutator was used as described above to make and break the battery circuit so as to increase the sensibility by giving a steady deflection of the galvanometer when the condition for balance was not fulfilled.

The mutual inductance M of two coils may be compared with the self-inductance L of a third coil by the following method, which is also due to Prof. C. Niven.* One of the mutually acting coils is included in the battery branch AB, Fig. 116, of a Wheatstone bridge,

Mutual Inductance of Two Coils compared with Self-Inductance of Third

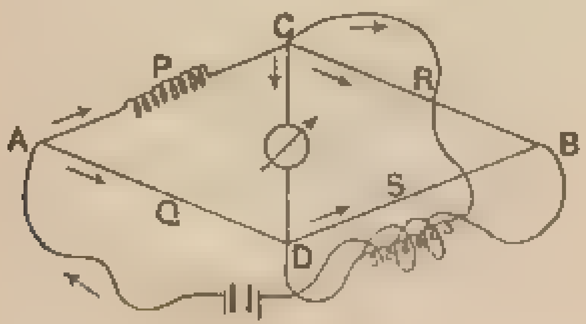


Fig. 116.

the other is placed as a shunt across the galvanometer branch CD. Balance is first obtained for steady currents, then the resistance, S', of the shunt is altered until there is no deflection of the needle at make and break of the battery circuit. Then

$$\frac{L}{M} = \frac{(P+Q)^2}{QS'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

We shall denote by P, Q, R, S, G, as before, the resistances of the four branches of the bridge and the galvanometer, by S', L', the resistance and inductance of the coil shunting the galvanometer branch, by Γ the inductance of the galvanometer, by $\dot{w}$, $\dot{x}$, $\dot{y}$, $\dot{z}$ the currents at any instant through the battery, the branch AC, the galvanometer, and S', and by γ the steady cur-

Theory of Method

* *Phil. Mag.*, Sept. 1887.

rent through the battery. From the galvanometer circuits $ACDA$, $CBDC$, we get the integral equations of currents

$$\left.\begin{aligned}(P+Q)x + Gy &= -Li_0 + Qu\\ (R+S)x - (R+S+G)y - (R+S)z &= Su\end{aligned}\right\} \quad . \quad . \qquad (61)$$

in which the inductances of the galvanometer and the coil which shunts the branch CD do not appear, since there is no steady current from C to D and the inductances of the other branches are supposed negligible.

The difference of potential between C and D is

$$\Gamma\dot{y} + G\dot{y} = M\dot{i} + L'\dot{z} + S'z$$

This integrated yields

$$Gy = M\gamma + S'z,$$

which converts the second equation of currents just found into

$$(R+S)x - \left(R+S+G+\frac{R+S}{S'}G\right)y = -M\frac{R+S}{S'}\gamma + Su \qquad (62)$$

Integral Flow through Galvanometer.

Eliminating x between (62) and the first of (61) with $Q\gamma/(P+Q)$ put for i_0 and using the relation $PS=QR$, we find

$$y = \gamma\frac{M\dfrac{(P+Q)^2}{S} - LQ}{(P+Q)\left\{P+Q+G\left(1+\dfrac{P}{R}+\dfrac{P+Q}{S'}\right)\right\}}$$

or since $\gamma = E/\{r + Q(P+R)/(P+Q)\}$, where r is the resistance of the battery branch AB, including coil and connections

$$y = E\frac{M\dfrac{(P+Q)^2}{S'} - LQ}{\left\{P+Q+G\left(1+\dfrac{P}{R}+\dfrac{P+Q}{S'}\right)\right\}\left\{r(P+Q)+Q(P+R)\right\}} \qquad (63)$$

Condition of Zero Integral Flow

The necessary and sufficient condition that y may be zero is thus

$$\frac{L}{M} = \frac{(P+Q)^2}{QS'}$$

which is (60). Hence when the resistances are so adjusted that there is no integral transient current in the galvanometer branch the inductances have this ratio.

Most Sensitive Arrangement of Bridge.

It is clear that since P is fixed the value of S' depends on that chosen for Q. To a certain extent S' is fixed and therefore also Q, since S' cannot be less than the resistance of the coil and connections used across CD. If P and Q be supposed both given, the best value of R to choose would be given by the equation

$$R^2 = \frac{\{QP + r(P+Q)\}\, GPS'}{GQ(S'+P+Q) + QS'(P+Q)} \quad . \quad . \quad (64)$$

Example of Method.

The following example is given by Prof. Niven. The field magnets of an old dynamo of the Ladd pattern were joined up in AC, and their self-inductance was compared with the mutual inductance of a pair of experimental coils. The resistance of that one of these coils which was placed in CD was 10·5 ohms, the resistance P of AC was 1·79 ohms, R was made equal to P, and Q was chosen 1000 ohms, so that S was also 1000 ohms. It was found that for balance an additional resistance of 167 ohms was required, making S' 177·5 ohms. Thus

$$\frac{L}{M} = \frac{(1001{\cdot}79)^2}{1000 \times 177{\cdot}5} = 5{\cdot}65$$

Comparison of an Inductance with a Resistance. Lord Rayleigh's Method.

The following method of determining a self-inductance in absolute measure, by comparing it with a resistance, has been used by Lord Rayleigh in his determination of the absolute value of the B.A. unit of resistance.* The method is originally due to Clerk Maxwell, and is described in his paper on "A Dynamical Theory of the Electromagnetic Field".† Four resistances, P, Q, R, S, are joined as four branches of a Wheatstone bridge, as shown in Fig. 119. The branch AC has self-inductance L but none of the others in-

* *Phil. Trans. R. S.*, Part II., 1882.

† *Phil. Trans. R. S.*, vol. clv., 1865; or Clerk Maxwell's *Collected Papers*, vol. i., p. 547.

ductance of any kind. A battery is placed in the branch AB, and a ballistic galvanometer in the branch CD.

Balance for steady currents is first obtained by depressing the battery key K_1, and a second or so afterwards the key K_2. Then K_2 is depressed first, and the angular deflection θ_1, produced by putting down K_1, is observed.

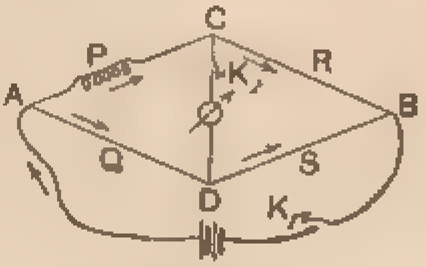


FIG. 117.

The balance for steady currents is now disturbed by altering the resistance P to $P + \delta P$, or Q to $Q + \delta Q$. We shall suppose that the latter change is made. The deflection θ_2, produced by the steady current which now flows through the galvanometer when both keys are put down, is read off and noted.

If $\dot{x}_1$, $\dot{x}_2$ be the steady currents which flow through the branches AC, AD respectively, after Q is changed to $Q + \delta Q$, and T be the period of oscillation of the needle, then it is shown below that, subject to correction for damping,

$$L = \delta Q \frac{\dot{x}_2}{\dot{x}_1} \frac{T}{\pi} \frac{\sin \frac{1}{2} \theta_1}{\tan \theta_2} \quad . \quad . \quad . \quad . \quad (65)$$

The ratio $\dot{x}_2/\dot{x}_1$ can be found as described below, and thus L can be calculated.

The secohmmeter can be applied to increase the sensibility of this method, and the arrangement of the apparatus is shown in Fig. 118. *BC* denotes the battery commutator, *GO* the galvanometer commutator. The arrows show the direction of rotation of each as seen from its side of the instrument. After the bridge has Use of Secohmmeter.

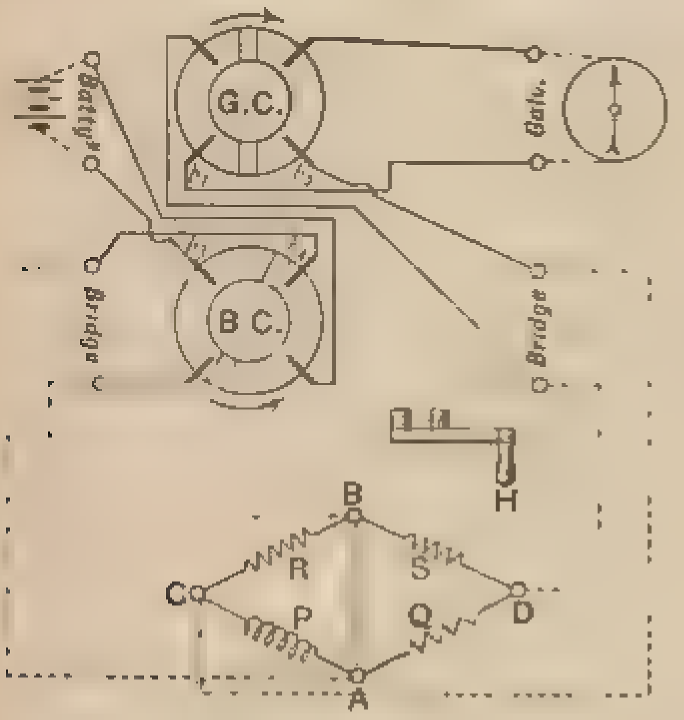


FIG. 118.

The continuous lines here represent permanent connections inside instrument the dotted lines temporary connections to bridge, &c

been balanced for steady currents, the instrument is rotated at a speed determined by a speed-measurer, and makes say n reversals per second. Let the steady deflection of the galvanometer needle be θ_1, then the uniform current equivalent to that producing the deflection is $H \tan \theta_1 / G$, where H is the field intensity acting on the needle, and G is the constant of the galvanometer, supposed to be a tangent instrument.

The secohmmeter is now stopped, and a steady current through the galvanometer is produced by altering Q to $Q + \delta Q$. Then it will be seen from the investigation below, that

$$L = \frac{\delta Q}{n} \frac{P}{Q} \frac{\tan \theta_1}{\tan \theta_2}$$

or

$$L = \frac{\delta Q}{n} \frac{P}{Q} \frac{\theta_1}{\theta_2} \qquad (65')$$

if the deflections are small.

Null Method by Secohmmeter

By first balancing for steady currents, then altering Q by a convenient amount δQ, and rotating the commutators at a proper speed, the induction current may be made to balance that due to the disturbance of balance so that no deflection is produced. When this is the case

$$L = k \frac{\delta Q}{n} \frac{P}{Q} \frac{\theta_1}{\theta_2} \qquad (65'')$$

if the angular deflections are small. Here k is a coefficient depending on the relative positions of the galvanometer and battery commutators, and may be determined once for all by determining the other quantities for a known self-inductance L. The galvanometer must not be reversed exactly or very nearly midway between two reversals of the battery, as the more nearly this arrangement is made, the smaller must be the value of L and the greater $\delta Q \,.\, P/Q$ for the necessary balance

Theory of Method.

The integral transient current through the galvanometer is easily found as follows. Let $\dot{x}$, $\dot{y}$, $\dot{z}$ be the currents in AC, through the galvanometer, and through the battery, at any instant, Γ, G, the self-inductance and resistance of the galvano-

meter, then from the circuits *ACDA*, *CBDC* (Fig. 117) we get the equations of currents

$$L\ddot{x} + P\dot{x} + \Gamma\ddot{y} + G\dot{y} - Q(\dot{u} - \dot{x}) = 0$$

$$R(\dot{x} - \dot{y}) - S(\dot{u} - \dot{x} + \dot{y}) - \Gamma\ddot{y} - G\dot{y} = 0.$$

These integrated over the whole interval of variation give with $\dot{x}_s$ put for the steady current in *AC*,

Integral of Varying Flow through Galvanometer.

$$\left.\begin{aligned}(P+Q)x + Gy &= Qu - L\dot{x}_s\\ (R+S)x - (R+S+G)y &= Su\end{aligned}\right\} \quad . \quad . \quad . \quad (66)$$

Hence

$$y = \frac{-L\dot{x}_s}{G\left(1+\frac{P}{R}\right) + P\left(1+\frac{S}{R}\right)} \quad . \quad . \quad . \quad . \quad (67)$$

Thus the flow through the galvanometer is the same as that due to an electromotive impulse, $L\dot{x}_s$ in *AC*, acting independently of the battery branch *AB*. For, any electromotive force e, thus acting, would give a current through the galvanometer of amount

$$\frac{e}{P+Q+\dfrac{G(R+S)}{G+R+S}}\,\frac{R+S}{G+R+S} = \frac{e}{G\left(1+\frac{P}{R}\right)+P\left(1+\frac{S}{R}\right)}.$$

The inductance of the galvanometer would not affect this result, and is therefore not introduced. Thus, if we put for the integral of e the value $L\dot{x}_s$, we get the result stated above.

Now if $\dot{z}_s$ denote the steady current through the branch *AD*, the steady currents through the galvanometer and the other branches of the bridge satisfy the equations (obtained from the circuits *ACDA*, *CBDA*, and the circuit *ACBA*, through the battery),

$$\left.\begin{aligned}P\dot{x}_s + G\dot{y}_s - Q'\dot{z}_s &= 0\\ R\dot{x}_s - (G+R+S)\dot{y}_s - S\dot{z}_s &= 0\\ (P+R+r)\dot{x}_s - R\dot{y}_s + r\dot{z}_s &= E\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (68)$$

where Q' denotes any value of the resistance of the branch *AD*, r the resistance and E the electromotive force of the battery.

Steady Current through Galvanometer.

Putting $Q' = Q + \delta Q$, and using the relation $SP = QR$, we get from these equations

$$\dot y_s = E\frac{R\delta Q}{\Delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

where Δ is the determinant of the system of equations (68).

But eliminating $\dot x_s$, $\dot y_s$, we find for the steady current $\dot z_s$ through the branch AD

$$\dot z_s = E\frac{GR + P(G + R + S)}{\Delta}.$$

Hence from (69)

$$\dot y_s = \frac{\dot z_s\delta Q}{G + \frac{P}{R}(G + R + S)}.$$

It may be noticed that an electromotive force $\dot z_s\delta Q$ in AD, acting as if the battery branch did not exist, would produce through the galvanometer a steady current of amount

$$\frac{\dot z_s\delta Q}{P + Q' + \frac{G(R+S)}{G+R+S}}\,\frac{R+S}{G+R+S} = \frac{\dot z_s\delta Q}{G + \frac{P}{R}(G+R+S) + \delta Q\frac{G+R+S}{R+S}}$$

which is nearly the same thing as $\dot y_s$ if δQ be small. It is to be carefully noticed here that $\dot z_s$ is the current in the branch AD *after* the resistance Q has been altered to $Q + \delta Q$.

By the theory of the ballistic galvanometer (p. 392 above) y is given by the equation

$$y = \frac{HT}{\pi G}\sin\tfrac{1}{2}\theta_1$$

subject to a correction for damping. Also

$$\dot y_s = \frac{H}{G}\tan\theta_2$$

Expression for L in Terms of Resistance.

so that $y/\dot y_s = L\dot x_s/\dot z_s\delta Q = T\sin\frac{1}{2}\theta_1/\pi\tan\theta_2$, or

$$L = \delta Q\frac{\dot z_s}{\dot x_s}\,\frac{T}{\pi}\,\frac{\sin\frac{1}{2}\theta_1}{\tan\theta_2}$$

which is equation (65).

Lord Rayleigh's Experiments.

In Lord Rayleigh's experiments the battery current was reversed to produce the induction-flow through the galvanometer; so that taking the deflection produced by reversal in each case we must use the ratio $i_t/2i_s$ in the above formula for L.

Arrangement of Coils.

Lord Rayleigh used for R and S two coils of ten units each taken from a resistance box, while P was a copper coil of resistance rather less than 24 ohms, and inductance L to be determined. A coil of 24 units taken from the same resistance box with a coil of 753 units (which was taken from an auxiliary box) placed in multiple arc with it, balanced P. The resistance P was thus $24 \times 753/777 = 23{\cdot}25869$, in units of the box.

Mode of Altering Q.

Q was altered by substituting 853 units from the auxiliary box for the 753 units used in multiple arc with the coil of 24 units. Thus Q was made 23·34322 units, and therefore δQ was ·08453 unit.

The battery current was reversed by a key placed in AB while the galvanometer branch was kept closed. Observations of θ_1, θ_2 were taken by means of telescope and scale in the ordinary manner; and were made as rapidly as possible, by properly manipulating the key, and opening and closing the galvanometer branch so as to stop the inductive deflections after the throw had been observed. The observer himself damped the vibrations of the needle by exciting temporarily at proper times a current in a coil for the purpose.

Method of Observing Induction Deflection by Reversal of Battery.

The induction throw was taken without waiting for the needle to come perfectly to rest, or arranging for perfect balance for steady currents. The amplitude of free swing was obtained by observing two successive elongations with the needle fairly quiet. Then the battery current was reversed as the needle passed through the position of equilibrium, and it was noted whether the induction throw was with or against the direction of free motion, and the four elongations after reversal were observed.

Correction of Throw for Change of Zero.

After reversal the zero for steady flow had of course shifted owing to imperfect balance, but the change gave a means of correcting the induction throw. Let a be double the true arc of deflection due to induction, a_0 the range of vibration from side to side just before reversal, and b the arc through which the zero had shifted, then at the moment after reversal the velocity which the needle had in consequence of free swing was numerically $\pi a_0/T$, in consequence of induction $\pi a/T$, and the displacement from the new zero was b. The velocity was thus $\pi(a \pm a_0)/T$.

If now s represent the displacement from the new zero at any subsequent time we have

$$s = A \sin\left(\frac{2\pi}{T} t - e\right)$$

where A and e are constants. Then

$$\frac{ds}{dt} = \frac{2\pi A}{T} \cos\left(\frac{2\pi}{T} t - e\right)$$

$$= \frac{2\pi A}{T} \cos e = \frac{\pi\,(a \pm a_0)}{T}$$

when $t=0$. Thus

$$A \cos e = \tfrac{1}{2}\,(a \pm a_0).$$

Again when $t=0$, $s=b$, and therefore

$$A \sin e = -b.$$

Hence we have

$$s = \tfrac{1}{2}\,(a \pm a_0) \sin \frac{2\pi}{T} t + b \cos \frac{2\pi}{T} t.$$

This represents a vibration of which the amplitude

$$A = \sqrt{\tfrac{1}{4}\,(a \pm a_0)^2 + b^2}$$

or, if b be small,

$$2A = a \pm a_0 + \frac{2b^2}{a}$$

so that

$$a = 2A \mp a_0 - \frac{2b^2}{a}$$

Corrected Value of Induction Deflection. where A was the observed arc of deflection. The correction given by the last term was very small. $2A$ was the arc between the two turning points immediately following the reversal. As a check readings of the two following turning points were also taken. The new zero was obtained from two successive elongations of the needle which were observed after the needle had nearly come to rest in its new position.

The next time the needle passed through the equilibrium position an induction throw in the opposite direction to the last was taken, and the four immediately following elongations observed.

Observations of Steady Current Deflection.

Readings were then taken as quickly as possible of the steady current deflection produced by changing the coil of 753 units to 853 units. Readings of three or four successive elongations were taken as soon as the amplitudes had become moderate. Then the galvanometer branch *CD* was opened, and the battery current was reversed while the needle was passing over to the other side of zero. When the needle had swung over, the galvanometer contact was restored, then four elongations were again observed. The arc between the two positions of equilibrium was thus twice the deflection due to the steady current produced by changing Q from 23·25869 to 23·34322 units.

A correction of course had to be made for the effect which would have been produced by reversing without changing Q. This was obtained from the observations of the effect of imperfect balance made before each induction throw, and any progressive change due to alteration of temperature was got rid of by using the mean of such observations made before and after a change from 753 to 853 units.

The following is a specimen set of observations. In the table E. P. stands for "equilibrium position," and I. T. for "induction throw."

Results of Observation.

Time of Observation	Position of Battery Key	Readings on Scale, and Deflections in Scale Divisions	
3 h. 36 m.	Left.	E. P. 264·4 I. T. 246·6	Res. 753 units.
3 h. 38 m.	Right.	E. P. 262·5 I. T. 2[illegible]5·9	
3 h. 40 m.	Right.	E. P. 182·3	Res. 853 units.
3 h. 41 m.	Left.	E. P. 344·7	
3 h. 44 m.	Left.	E. P. 264·4 I. T. 245·7	Res. 753 units.
3 h. 45 m.	Right.	E. P. 263·1 I. T. 245·6	

Reduction of Results of Observations.

In the first set of these results the difference 1·9 between 264·4 and 262·5 was due to imperfection of balance, in the second set the difference was 1·3. The mean of these, 1·6, subtracted from 162·4 gave 160·8 as the deflection produced by replacing 753 units by 853 and reversing, corrected for imperfection of balance.

Thus the ratio of the two deflections obtained from this specimen set of observations was 245·9/160·8 = 1·529. Two sets, each of four similar observations, the second set made with the galvanometer reversed, gave each the mean value 1·5310 for this ratio, so that reversing the galvanometer produced no effect.

Calling D the distance of the mirror from the scale, $2A$ the induction deflection, $2B$ the deflection produced by reversing the battery current when balance is disturbed by the addition of 100 units to the 753, all three quantities being expressed in terms of the same unit of length, we have

$$\tan 2\theta_1 = \frac{A}{D}, \quad \tan 2\theta_2 = \frac{B}{D}$$

which give by successive approximation

$$\frac{2 \sin \frac{1}{2} \theta_1}{\tan \theta_2} = \frac{A}{B} \, \frac{1 - \frac{11}{32} \frac{A^2}{D^2}}{1 - \frac{1}{4} \frac{B^2}{D^2}}$$

or since $A = 122·5$, $B = 80$, and $D = 2180$,

$$\frac{2 \sin \frac{1}{2}\theta_1}{\tan \theta_2} = ·99925 \frac{A}{B} = ·99925 \times 1·5310.$$

Correction of Induction Deflection for ing.

Separate determinations of the logarithmic decrement gave $\lambda = ·0142$, and the period T was found to be 23·386 seconds. Since the effect of damping was to diminish the distances from zero at the first and second elongations by the fractions $\frac{1}{2}\lambda$, $\frac{3}{2}\lambda$ of their proper amount, the difference between these distances can be corrected by multiplying by the factor $1 + \lambda$. It is sufficient to apply this factor to the value of $2 \sin \frac{1}{2}\theta_1 / \tan \theta_2$.

Thus the equation for L becomes

$$L = \delta Q \frac{t_1}{t_2} \frac{T}{4\pi} ·99925 \frac{A}{B}(1 + \lambda) \quad . \quad . \quad . \quad . \quad (73)$$

The resistance of the galvanometer was 80 units, and calculation showed that the current through it might be neglected

in estimating the ratio $i_s : i_a$. The resistance of the battery being low, the difference of potential between A and B was taken as given. Calling it V we have

$$i_s = V/(10 + 23{\cdot}25869),\ i_a = V/(10 + 23{\cdot}34322),$$

so that

$$\frac{i_a}{i_s} = \frac{10 + 23{\cdot}25869}{10 + 23{\cdot}34322}$$

Using then these data with the value ·08453 × ·987 ohm, or ·08453 × ·987 × 10^9 C. G. S. for δQ obtained by regarding 1 B. A. unit as ·987 ohm, we get

Final Results

$$L = 2{\cdot}4028 \times 10^8$$

in ordinary electromagnetic C.G.S. units, that is, in cms.*

At the temperature of the room the resistances given by the boxes were not exactly multiples of the B. A. unit, and the resistance of 853 units had to be increased by fully one part in a thousand to give the necessary correction. Thus δQ was greater than the value given above by this fraction. Thus finally

$$L = 2{\cdot}4052 \times 10^8, \text{ in cms.}$$

Calculation from the specification of the coil gave

$$L = 2{\cdot}400 \times 10^8, \text{ in cms.}$$

about 1 in 500 less. In Lord Rayleigh's judgment the former value was just as likely to be correct.

Joubert's Method of Measuring Self-Inductance.

A self-inductance may also be compared with a resistance by the following method due to M. Joubert. A circuit is made up of the coil the inductance of which is to be determined, and a non-inductive resistance. An alternating machine giving a suitable electromotive force as nearly as possible following the simple harmonic law is included, and the mean square of the difference of potential between the terminals is

* See next Chapter.

compared by means of an electrometer with that existing between the terminals of the non-inductive resistance. Denoting the mean squares of these differences, respectively, by $\overline{V_1}^2$, $\overline{V_2}^2$, and the resistances of the corresponding coils by R_1, R_2, we have

$$\frac{\overline{V_1}^2}{\overline{V_2}^2} = \frac{R_1^2 + n^2L^2}{R_2^2} \quad . \quad . \quad . \quad . \quad . \quad (70)$$

where $n = 4\pi/T$, T being the complete period of the alternating current. This equation gives

$$L = \frac{R_1}{n}\left(\frac{R_2^2}{R_1^2}\frac{\overline{V_1}^2}{\overline{V_2}^2} - 1\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (71)$$

The value of n can be found of course from the speed of the machine, and the number of alternations in each turn.

To find the ratio $\overline{V_1}^2/\overline{V_2}^2$ the electrometer must be used idiostatically as explained in Vol. I. p. 299, that is, one terminal is connected to one pair of quadrants if the instrument is a quadrant electrometer; or to the stationary electrified system which acts on the movable system or indicator, while the other terminal is attached to the needle or indicator. Then the mean square of the difference of potential between the terminals will be proportional to the deflection if small, or if the needle is brought back to a sighted zero position, will be proportional to the couple required to keep it in that position. Sir William Thomson's multicellular electrostatic voltmeter * is well adapted for this measurement.

* See the Author's Smaller Treatise, p. 142.

Theory of Method.

To prove the formulas stated above let r be the part of the resistance which does not depend on the coils used for the comparison, $E \sin nt$, the electromotive force in the circuit at any instant, and $\dot{x}$ the current at that instant. Then if $L + L'$ is the total inductance in the circuit

$$(L + L')\ddot{x} + (R_1 + R_2 + r)\dot{x} = E \sin nt.$$

The part $L\ddot{x} + R_1\dot{x}$ is the difference of potential then existing between the terminals of the coil that is being tested, $R_2\dot{x}$ is that between the terminals of the non-inductive coil. We may write, therefore, if v_1, v_2, be constants

$$\left.\begin{aligned} L\ddot{x} + R_1\dot{x} &= v_1 \sin nt \\ R_2\dot{x} &= v_2 \sin nt \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (72)$$

The complete solution of the first of these equations is

$$\left.\begin{aligned} \dot{x} &= A\epsilon^{-\frac{R_1}{L}t} + \frac{v_1}{\sqrt{R_1^2 + n^2L^2}} \cos(nt - e) \\ \text{where} \quad & \\ \tan e &= \frac{R_1}{nL} \end{aligned}\right\} \quad . \quad . \quad . \quad (73)$$

The first term on the right dies out in a short time, and has no further influence if the machine works regularly, and so

$$\dot{x} = \frac{v_1}{\sqrt{R_1^2 + n^2L^2}} \cos(nt - e).$$

By this result and the second of (72)

$$R_2\dot{x} = \frac{v_1 R_2}{\sqrt{R_1^2 + n^2L^2}} \cos(nt - e) = v_2 \sin nt$$

and therefore

$$v_1^2 \cos^2(nt - e) = \frac{R_1^2 + n^2L^2}{R_2^2} v_2^2 \sin^2 nt.$$

Hence, integrating over a complete period, we find

$$\frac{\overline{V_1}^2}{\overline{V_2}^2} = \frac{R_1^2 + n^2L^2}{R^2}$$

which is (70), and the rest follows as above.

Comparison of Self-Inductance with Capacity of a Condenser.

Maxwell also showed how to compare the inductance of a coil with the capacity of a condenser, and his method has since been modified by various experimenters so as to obviate the necessity for successive adjustments which it involves. As originally given the method consisted in placing the coil in one branch of a Wheatstone bridge, as DB, Fig. 119, while the plates of the condenser were attached directly at AC. Balance for steady currents is first obtained and is not affected by the condenser; then the resistances are altered until no inductive flow through the galvanometer is produced by making or breaking the battery circuit. If C be the capacity of the condenser, P, S the resistances of the branches AC, DB, the relation fulfilled when balance is thus obtained is

$$L = PSC. \qquad (74)$$

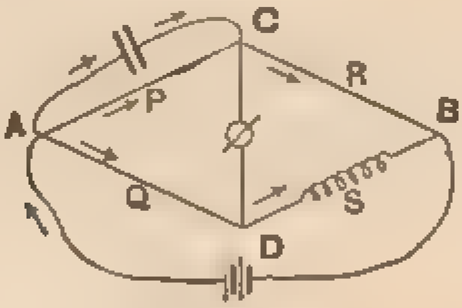


FIG. 119.

Theory of Method

Let as before P, Q, R, S denote the resistances of AC, AD, CB, DB, L the inductance in the branch DB, and put C for the capacity of the condenser. Let further for any instant $\dot z$ denote the current along AC, $\dot x - \dot z$ the current charging the condenser, $\dot y$ the current from A to D, and ξ, η the potentials at C and D. Suppose that balance for steady currents is first obtained so that $PS = QR$, then in order that at the instant in question ξ may be equal to η the conditions

$$P\dot z = Q\dot y \qquad (75)$$

$$L\ddot y + S\dot y = R\dot z \qquad (76)$$

must hold. But $P\dot{z}$ is the difference of potential between A and C, and may be taken as that between the plates of the condenser. Hence the charge of the condenser is $CP\dot{z}$, and since $\dot{x}-\dot{z}$ is the rate of increase of this charge we have

$$\dot{x}-\dot{z}=CP\ddot{z}=CQ\ddot{y}.$$

This with (75) converts (76) into

$$L\ddot{y}+S\dot{y}=\frac{RQ}{P}\dot{y}+RCQ\ddot{y}\,. \quad . \quad . \quad . \quad . \quad (77)$$

which if ξ is always to be equal to η must hold for all values of $\dot{y}$ and $\ddot{y}$. But $PS-RQ=0$; hence we must have also

$$L=PSC \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

and S and P must be chosen so as to fulfil this condition if the current through the galvanometer is always to be zero.

A series of successive adjustments is thus necessary before the proper values of S and P, and balance for steady currents are obtained. Mr. E. C. Rimington * has shown how these adjustments may be avoided by Rimington's Modification of Method.

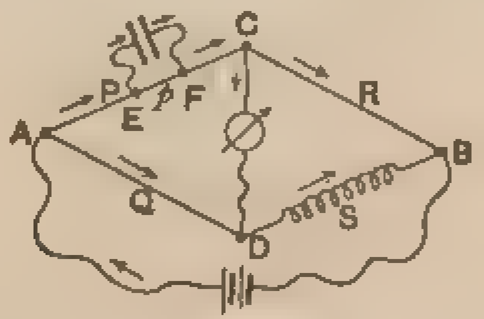


FIG. 120.

a very simple modification of the method. The balance for steady currents having been obtained as before, the condenser is applied at two points E, F, in AC (Fig. 120), including between them a resistance p ($< P$) such that

* *Phil. Mag.* July, 1887.

with the inductance L in DB no deflection of the galvanometer needle takes place when the battery key is depressed or raised. The resistance p may be taken from a resistance slide the whole (or variable part) of which is included in AC, or preferably two slides in series may be used so as to give two adjustable sliding contact pieces to which to attach the plates of the condenser. The galvanometer needle should have sufficient moment of inertia to enable the whole inductive action to begin and end before the needle has sensibly moved, for the effect of the condenser AC, which is charged by the current from A to E, is to delay the rise of the potential at C to its final value after the battery key is put down, while the inductance L in DB produces a similar effect on the rise of the potential at C; hence if the needle were not sufficiently ballistic it might show a deflection due to a difference in the rate of variation in the two cases, although the time-integral of the current through the galvanometer were really zero. The inductance is given by the equation

$$L = Cp^2 \frac{S}{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

Theory of Modified Method. Writing down the equations of currents for the circuits $ACDA$, $CBDC$, putting $\dot{x}$ for the current in AE and FC, $\dot{z}$ for the current in EF, using the same notation as before for the other quantities, and integrating over the time interval from the instant before completion of the battery circuit until the steady state has been attained, we find by (6)

$$\left.\begin{aligned} (P+Q)x \qquad\qquad\quad + Gy &= Qu + Cp^2\dot{x}_1 \\ (R+S)x-(G+R+S)y &= Su + L(\gamma-\dot{x}_1) \end{aligned}\right\} \quad . \quad . \quad (80)$$

where γ, $\dot{x}_1$, denote the steady currents in the battery and in

the branch AC. Solving for y and putting $y - \dot{x}_1 = \dot{x}_4 P/Q$, we find

$$y = \frac{(R+S)(Cp^2S - LP)\dot{x}_4}{S\{G(R+S) + (G+R+S)(P+Q)\}} \quad . \quad . \quad (81)$$

Thus the necessary and sufficient condition that there should be no integral flow through the galvanometer is

Condition for Zero Integral Flow.

$$L = Cp^2 \frac{S}{P},$$

as already stated.

If $p = P$ this gives the result already obtained for the case originally considered by Maxwell.

It ought to be noticed here that precisely the same equation may be obtained by integrating, in the same way, over the interval at break from the steady state to zero current in each conductor, so that the test may be repeated at breaking the circuit.

We may now investigate the most sensitive arrangement of the bridge. In general S is given in magnitude, and p, which must of course be less than P, will in most cases be some convenient resistance depending on the apparatus available, so that P may be regarded as given. Hence we have to choose the value of R (and that of Q will follow) so that y may for some chosen value of p be a maximum. By (81) and the equation

Most Sensitive Arrangement of Bridge.

$$\dot{x}_4 = \frac{SE}{r(R+S) + S(P+R)}$$

where E is the electromotive force of the battery, and r the resistance of the battery and the wires connecting it to A, B, we get easily

$$y = \frac{(Cp^2S - LP)E}{\left\{G\left(1+\frac{P}{R}\right) + P\left(1+\frac{S}{R}\right)\right\}\left\{r(R+S) + S(P+R)\right\}} \qquad (82)$$

The numerator of this expression does not vary: hence calling the denominator D, calculating dD/dR, and equating to zero, we find after reduction

$$R^2 = \frac{SP(G+S)(r+P)}{(G+P)(r+S)} \quad . \quad . \quad . \quad . \quad . \quad (83)$$

which gives the best value of R if that of G is given.

If however there is a choice of similar galvanometer bobbins of different resistances, then as before (p. 473) we must substitute for D a value $D' = D/\sqrt{G}$, calculate dD'/dG, and equate the result also to zero. This gives another equation for G and R, viz.

$$G = \frac{P(R+S)}{P+R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

From (83) and (84) as simultaneous equations the values of G and R are to be found.

If p is at the disposal of the experimenter and can be varied by small steps, the best arrangement is that for which when y is almost zero a given small change in p gives a maximum change in y. Hence if possible we have to arrange so that dy/dp may be a maximum when $y=0$. The conditions for this however are so complicated as to be unserviceable.

Anderson's Ballistic Method.

Professor Anderson * has also given the following simple ballistic method of comparing the capacity of a condenser with an inductance A bridge is made up as before of four conductors, and a condenser and galvanometer are arranged as in Fig. 121, so that by

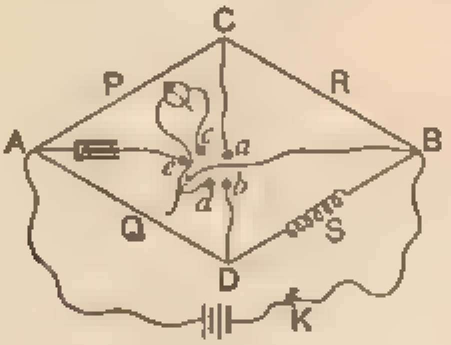


Fig. 121.

means of mercury cups the galvanometer can be connected either to CD by the cups a, b, c, d, or in series with the condenser in the branch AB by the cups

* *Phil. Mag.* April, 1891.

c, d, e, f. A rocking key is conveniently made to effect either of these connections at a single operation. A coil of inductance L is placed in AC, all the other branches with the exception of the galvanometer are destitute of inductance.

Balance for steady currents is first obtained with the galvanometer in CD. Then when the key K is depressed or raised an inductive flow of integral amount y passes through the galvanometer. If x_s is the steady current in DB, the value of y is (p. 481) given by

$$y = \frac{Lx_s}{G\left(1+\frac{S}{Q}\right)+S\left(1+\frac{P}{Q}\right)} \quad . \quad . \quad . \quad (85)$$

The deflection θ_1 produced by this is noted.

By means of the rocking key the galvanometer is joined in series with the condenser between the points A and B, so that the plates of the condenser are charged to a difference of potential $x_s(Q+S)$. If C be the capacity of the condenser a quantity of electricity $Cx_s(Q+S)$ passes through the galvanometer. The resulting deflection θ_2 is observed.

We have then by the theory of the ballistic galvanometer and (85)

$$L = C(Q+S)\left\{G\left(1+\frac{S}{Q}\right)+S\left(1+\frac{P}{Q}\right)\right\}\frac{\sin\frac{1}{2}\theta_1}{\sin\frac{1}{2}\theta_2}. \quad (86)$$

The following are the details of an actual measurement made by the author of the method. A coil of mean radius 20·9 cms. wound with 278 turns of wire in a groove of breadth 1·894 cms. and depth 1·116 cm., was placed in AC. The galvanometer was an ordinary reflecting instrument of resistance 164·8 ohms, with Practical Example of Method

its period made as long as possible by means of a controlling magnet. A non-inductive resistance of 100 ohms was added to the coil, and P and R were each 10 ohms. Balance was obtained by making S 150·51 ohms. The mean results of several readings agreeing well together were

Deflection due to induction . . .	43·208 divisions.
Deflection due to charge of condenser of ·5 microfarad	46·125 ,,
Deflection due to charge of condenser of ·45 microfarad	41·875 ,,

By interpolation it was found from these results that a condenser of ·4657 microfarad capacity would just give a deflection of 43·208 divisions. Thus in C.G.S. units *

$$L = \cdot 4657 \times 10^{-15} \times 2 \times 150{\cdot}51 \times (329{\cdot}6 + 150{\cdot}51 + 10) \times 10^{18}$$

$$= \cdot 0687 \times 10^{9}.$$

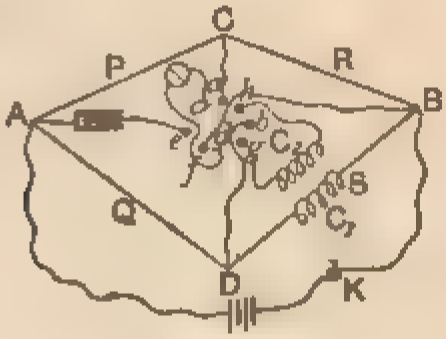


Fig. 122.

Method Applied to Compare Mutual Inductance and Capacity.

To determine a mutual inductance the method is used thus: One coil, C_1, of the mutually influencing pair is joined in DB as before, the other, C_2, has its terminals joined to a pair of mercury cups g, h, which are arranged so that a rocking-key can put the galvano-

* See next chapter for the specification of Practical Units. A microfarad is 10^{-15} C.G.S. units of capacity, and an ohm 10^{9} C.G.S. units of resistance.

meter between A and B, or between the cups g, h, so as to connect the terminals of the coil.

Balance for steady currents having been obtained as before, the terminals of the galvanometer are connected to g, h, and the battery circuit is completed or broken. Calling θ_3 the deflection produced and denoting by θ_1, θ_2, as before, the deflections obtained by operating with the coil C, as already described (p. 495), we have

$$M = C(Q+S)(r_2+G)\frac{\sin\frac{1}{2}\theta_3}{\sin\frac{1}{2}\theta_2} . \quad . \quad . \quad (87)$$

and

$$\frac{M}{L} = \frac{r_2+G}{G\left(1+\frac{S}{Q}\right)+S\left(1+\frac{P}{Q}\right)}\frac{\sin\frac{1}{2}\theta_3}{\sin\frac{1}{2}\theta_1} . \quad . \quad (88)$$

Theory of Method.

The inductive electromotive force at any instant in the coil C_2 is $M\dot{x}$, hence the integral electromotive force is Mx_0. The whole quantity of electricity which flows through the galvanometer is thus $Mx_0(r_2+G)$ where r_2 is the resistance of the coil C_2. But the quantity of electricity which passes when the throw θ_2 is produced is $Cx_0(Q+S)$. Hence we get (87), and combining (87) with (86) we get (88)

Practical Example of Method.

As an example Professor Anderson gives the following:—$Q=S=1{\cdot}003$ ohm, the resistance of the coil C_1; $r_2=157{\cdot}7$ ohms, $G=164{\cdot}8$ ohms, $C=1$ microfarad, θ_3, $\theta_2=72$ and 5 scale divisions respectively. Hence roughly, in C.G.S. units

$$\begin{aligned} M &= 10^5\times 2{\cdot}006\times 322{\cdot}5\times 14{\cdot}4 \\ &= 9315864 \end{aligned}$$

Method by Differential Galvanometer.

Professor Niven* has shown how to compare the inductance of a coil with the capacity of a condenser by means of a differential galvanometer. A circuit is made up as shown in Fig. 123, of one coil of the

* *Phil. Mag.* Sept. 1887.

differential galvanometer, the coil (of inductance L and resistance R_1) to be compared, an additional resistance in the branch AE and the battery B. A corresponding circuit is arranged with the other coil of the galvanometer, a non-inductive resistance R_2, an additional resistance in the branch AF, and the battery as before, so that the battery serves both circuits as shown in the Figure. After balance for steady currents has been obtained by adjusting the additional resistances, the

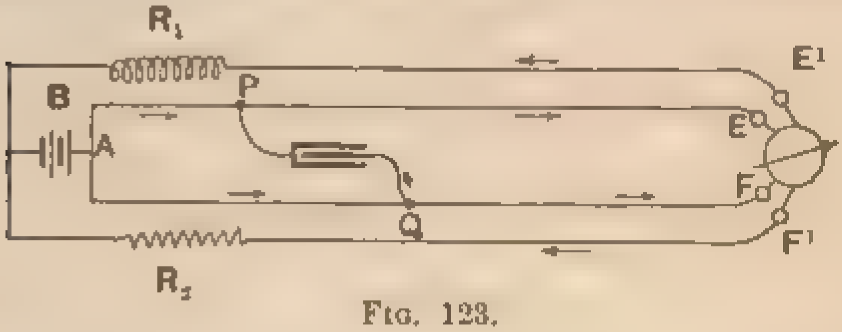


FIG. 123.

condenser is joined across the two branches AE, AF, and the terminals shifted until no deflection is produced when the battery-key is depressed, or raised, the circuits having been otherwise completed previously. When this is the case the following condition is fulfilled

$$L = C\,(R_2'^2 - R_1'^2). \quad . \quad . \quad . \quad . \quad (89)$$

where R'_1, R'_2, are the resistances from A to P and Q respectively (see Fig. 123).

Theory of Method. We shall suppose the coils of the galvanometer exactly equal for equal currents in magnetic effect on the needle, and that each has the same resistance G. Clearly, for balance with steady currents, the resistance of each circuit must be the same. Denoting therefore by R the resistance in each circuit, exclusive of the battery resistance, r, and the resistance G of the galvanometer coil, and putting E for the electromotive force of the

battery, we have for the steady current γ through either of the galvanometer coils $\gamma(R + G) + 2\gamma r = E$, or

$$\gamma = \frac{E}{R + G + 2r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

Let PQ be the points at which the terminals of the condenser are attached, R'_1 denote the resistance from A to P, R''_1, that from P to the nearest galvanometer terminal, R'_2, R''_2, the resistances from A to Q, and from Q to the galvanometer. Γ the inductance of each galvanometer coil, M their mutual inductance $\dot x - \dot z$ the current from A to P, $\dot y + \dot z$ that from A to Q, and $\dot z$ the current from Q charging the condenser. The equations of currents obtained from the two circuits $AEGE'A$, $AFGF'A$, are (since $R_1 + R'_1 + R''_1 = R_2 + R'_2 + R''_2 = R$).

$$(L + \Gamma)\ddot x + M\ddot y + (R + G + r)\dot x + r\dot y - R'_1\dot z = E$$

$$M\ddot x + \Gamma\ddot y + r\dot x + (R + G + r)\dot y + R'_2\dot z = E$$

Integrating these from before make to the steady state, putting γ for the steady current, and subtracting we find

Effective Integral Flow Producing Deflection.

$$(R + G)(x - y) + L\gamma - (R'_1 + R'_2)z = 0 \quad . \quad . \quad (91)$$

But the final charge of the condenser is $C(R'_2 - R'_1)\gamma$ if C denote its capacity, so that

$$z = C(R'_2 - R'_1)\gamma.$$

Substituting in the last equation we get

$$x - y = \gamma \frac{C(R'^2_2 - R'^2_1) - L}{R + G}$$

or

$$x - y = E \frac{C(R'^2_2 - R'^2_1) - L}{(R + G)(R + G + 2r)} \quad . \quad . \quad . \quad (92)$$

by (90).

If no deflection of the galvanometer needle takes place x must be equal to y, and for this the necessary and sufficient condition is

$$L = C(R'^2_2 - R'^2_1)$$

as already stated above in (89).

Most Sensitive Arrangement. With regard to the sensibility of the arrangement it is to be observed that R_1 is given, being the resistance of the coil to be compared, and in general G also is given, so that all that can be done to make the arrangement sensitive is to keep down the value of the resistance additional to R_1.

If the resistance of the battery is negligible and the galvanometer bobbins be a matter of choice, the best arrangement is to make the additional resistance as small as possible, and make $G = R$.

If the galvanometer coils be each shunted by a wire of resistance S the resistance of each galvanometer bobbin will become $GS/(G + S)$, which we denote by G', and thus, if the inductance of each shunt is the same, takes the place of G in (92). The integral flow through the coils is then $Sx/(G + S)$ for one, and $Sy/(G + S)$ for the other. Hence the total flow affecting the needle is $S(x - y)/(G + S)$, or $(x - y)\, G'/G$. But we now have

$$x - y = E\frac{C(R'^2_2 - R'^2_1) - L}{(R + G')(R + G' + 2r)} \quad . \quad . \quad . \quad (93)$$

Hence in order that $(x - y)\, G'/G$ may be a maximum, we must make $(R + G')(R + G' + 2r)/G'$ a minimum. Differentiating with respect to G' we find that the condition for a minimum is

$$G'^2 = R(R + 2r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

Thus if the galvanometer have a high resistance so that the deflections are small, an improvement can be effected by shunting down each coil of the instrument to an effective resistance given by this equation.

Anderson's Null Method. A modification of Maxwell's method which has the advantage of being a Null method, and therefore of permitting a telephone to be used instead of a galvanometer has been given by Prof. A. Anderson.* The arrangement of resistances is the same as before, but the condenser instead of being placed between A and C is placed between A and a point E on CD (Fig. 124). The galvanometer (or telephone) is supposed included

* *Phil. Mag.* April, 1891.

in the part ED of CD, and the resistance, g say, of CE, is varied until no deflection of the galvanometer needle is produced by making or breaking the battery circuit.

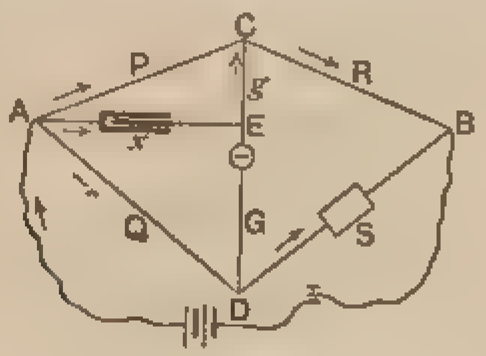


Fig. 124.

Let the resistance of ED be denoted by G, the currents through the galvanometer (from E to D) and to the condenser by $\dot{y}$, $\dot{z}$, so that the current from E to C is $\dot{z} - \dot{y}$. Thus from the circuits $ACDA$, $CBDA$ by integrating over the interval of variation, and using the value $Q\gamma/(P + Q)$ for $\dot{x}$, the steady current in AC, and $CP\dot{x}$, for the final charge z of the condenser, we get if the inductances of the other arms of the bridge are negligible **Theory.**

$$\left.\begin{aligned} (P+Q)x+(G+g)y &= -\frac{Q-g}{P+Q}CPQ\gamma+Qu \\ (R+S)x-(R+S+G+g)y &= \frac{P\gamma}{P+Q}\{L-CQ(R+S+g)\}+Su \end{aligned}\right\} \quad (95)$$

Eliminating x we find

$$y = \gamma \frac{P[C\{RQ+g(Q+S)\}-L]}{(R+S)(G+g)+(R+S+G+g)(P+Q)} \quad . \quad (96)$$

The value of y is zero if the numerator vanish, that is if

$$L = C\{RQ+g(G+S)\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

If $g = 0$ we fall back on Maxwell's solution, viz.

$$L = CRQ = CPS. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

Condition for Null Method.

That this is the necessary condition that the method may be a null one may be seen in the following manner. Whatever be the conductor between A and E the difference of potential between A and E is $P\dot{x} + g(\dot{y} - \dot{z})$, while that between A and D is $Q(\dot{u} - \dot{x} - \dot{z})$. If there is no difference of potential between E and D, $\dot{y} = 0$, and we have $P\dot{x} - g\dot{z} = Q(\dot{u} - \dot{x} - \dot{z})$. Integrating from just before the completion of the circuit to any instant during the interval of variation we find

$$Px - gz = Q(u - x - z) \quad . \quad . \quad . \quad . \quad . \quad (99)$$

Also from the branches ECB, DB we get in like manner

$$R(x + z) + gz = S(u - x - z) + L(\dot{u} - \dot{x} - \dot{z})$$

But by (99) the last equation may be written

$$Rx + (g + R)z - \frac{L}{Q}(P\dot{x} - g\dot{z}) = S(u - x - z) \quad . \quad . \quad (100)$$

Equation (99) multiplied by S and subtracted from the last equation multiplied by Q gives, since $PS = QR$,

$$\{QR + g(Q + S)\}z - L(P\dot{x} - g\dot{z}) = 0,$$

and since $P\dot{x} = z/C + g\dot{z}$, this is

$$C\{QR + g(Q + S)\}z - Lz = 0.$$

Hence

$$L = C\{QR + g(Q + S)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

That, conversely, the difference of potential between E and D is zero if this condition is fulfilled can be seen as at p. 460, from the consideration that otherwise there would be more than one solution of the problem of flow of electricity in the given network between A and B.

Most Sensitive Arrangement.

Returning to (96) putting for γ its value

$$E / \{r + S(P + R) / (R + S)\}$$

we write the equation in the form

$$y = E\frac{P[C\{RQ + g(Q + S)\} - L]}{\left\{Q + g + (R + S + Q + g)\dfrac{P}{R}\right\}\{r(R + S) + S(P + R)\}} \quad (102)$$

For sensitiveness a given change in g the adjustable resistance must produce a maximum change in y when y is nearly zero, that is dy/dg must be a maximum when $y=0$. We may neglect in all practical cases r, the resistance of the battery so that we have

$$\frac{dy}{dg}_{y=0} = \frac{CP(Q+S)E}{\left\{G+g+(R+S+G+g)\frac{P}{R}\right\}S(P+R)}.$$

But since

$$\frac{P(Q+S)}{S(P+R)} = \frac{P(Q+S)}{R(Q+S)} = \frac{P}{R}$$

this equation may be written

$$\frac{dy}{dg}_{y=0} = \frac{CE}{\frac{R}{P}(G+g)+R+S+G+g} \quad . \quad . \quad (103)$$

Hence in order that the denominator may be small we must take R and g small and P large, and therefore Q also large.

Comparison of Mutual Inductance and Capacity: Carey Foster's Method.

A method of comparing a coefficient of mutual induction with the capacity of a condenser has been given by Prof. Carey Foster.* It is based on the following

FIG. 125.

considerations. Let the two coils C_1, C_2, the mutual inductance for which is required be given in position as in Fig. 125, and be joined, one, C_1, through a battery, a coil of resistance R_1, a make and break key K, and the other, C_2, as a secondary circuit through a galvano-

* *Phil. Mag.* Feb. 1887.

meter G. Then if R_2 be the resistance of the secondary circuit, M the mutual inductance of the two coils, the whole quantity of electricity which flows through the secondary when a steady current of strength γ is produced or annulled in the primary is $M\gamma/R_2$.

Basis of Method

Again if the resistance coil in the circuit of C_1 have its terminals connected to a condenser of capacity C, (Fig 126) and the primary circuit be made or broken the quantity of electricity which traverses the galvanometer G is $CR_1\gamma$. Thus if the same deflection as before is obtained we have

$$M = CR_1R_2 \quad . \quad . \quad . \quad . \quad . \quad (104)$$

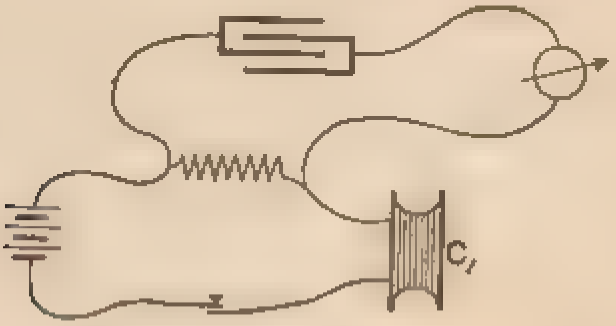


Fig. 126.

If however deflections are obtained indicating currents γ_1, γ_2, in the two cases, then

$$M = CR_1 R_2 \frac{\gamma_1}{\gamma_2} \quad . \quad . \quad . \quad . \quad . \quad (105)$$

Final Arrangement.

Now let a combination of these two arrangements be made as shown in Fig. 127, including a resistance box in the secondary circuit to enable the resistance R_2 of that circuit between the points A and E, to be varied at pleasure Then let the resistances R_1 (in the primary between the terminals of the condenser), and R_2 be

varied until on making or breaking the battery circuit no deflection is produced. When this is the case the integral flow through the galvanometer due to the charging of the condenser (that is the charge of the condenser) is exactly equal and opposite to that due to the induction current in the secondary circuit Thus noticing that the inductance in C_2 cannot affect the integral flow through it we see that $CR_1\gamma = M\gamma/R_2$, or

$$M = CR_1 R_2 \quad . \quad . \quad . \quad . \quad . \quad (106)$$

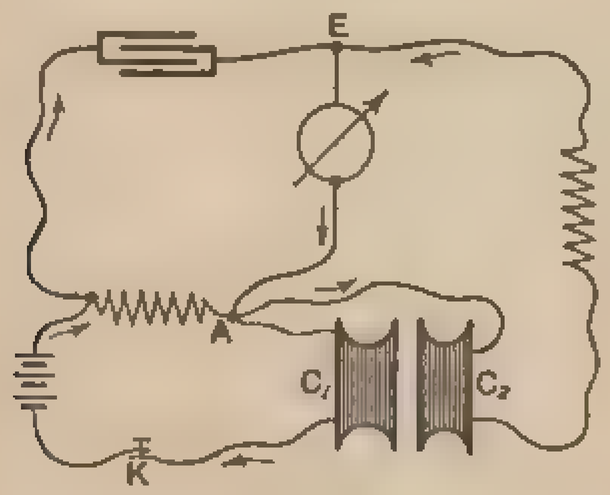


FIG. 127.

We can easily find the most sensitive arrangement for the experiment. In the first place it is to be noticed that the resistance (R'_1 say) other than R_1 in the primary circuit depends on the primary coil and the battery and is to be taken as fixed. We shall regard the galvanometer bobbin (1) as given (2, as a matter of choice from similar bobbins of different resistances. — **Most Sensitive Arrangement**

Let us suppose that the potential at A is not equal to that at E. Then putting $\dot{i}$, $\dot{x}$ for the currents in the primary and secondary, $\dot{y}$ for the current through the galvanometer, Γ for the inductance and G for the resistance of the galvanometer bobbin, we get from the circuit AC_2EA (Fig. 127) the equation $L\ddot{x} + M\ddot{i} + R_2\dot{x} + \Gamma\ddot{y} + G\dot{y} = 0$. This gives the integral equation

$$R_2 x + Gy = M\gamma.$$

Integral Flow through Galvanometer.

Further we have for the total charge of the condenser

$$-x+y=CR_1\gamma.$$

Solving for y from these we find

$$y=\frac{(CR_1R_2-M)\gamma}{G+R_2}$$

or since $\gamma=E/(R_1+R'_1)$ where E is the electromotive force of the battery

$$y=E\frac{CR_1R_2-M}{(G+R_2)(R_1+R'_1)}, \quad . \quad . \quad . \quad . \quad . \quad (107)$$

which gives the same condition as before that y may be zero.

In order that y may be a maximum the value of the denominator must be a minimum. Calling it D, and noting that R_1, R_2 only vary, and are connected by the relation $R_1R_2=eM/C$ where e is a small quantity we find

$$\frac{dD}{dR_1}=G+R_2+(R_1+R'_1)\frac{dR_2}{dR_1}=0$$

$$R_1\frac{dR_2}{dR_1}+R_2=0$$

Conditions of Maximum Sensibility.

Eliminating dR_2/dR_1 we get as the required condition of maximum sensitiveness with a given galvanometer

$$\frac{R_2}{R_1}=\frac{G}{R'_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

If the galvanometer bobbin is also at our disposal we have instead of the value of D found above to use

$$D'=D/\sqrt{G}=(R_1+R'_1)(\sqrt{G}+R_2/\sqrt{G}).$$

This gives in addition to (108)

$$\frac{dD'}{dG}=\tfrac{1}{2}(R_1+R'_1)\left(\frac{1}{\sqrt{G}}-\frac{R_2}{G^{\frac{3}{2}}}\right)=0,$$

$$G=R_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

Thus we have in the latter case as the conditions for maximum sensibility

$$R_1 = R_2 = G. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

If it can be arranged to maintain the two points A, E always at the same potential, we may use a telephone instead of a galvanometer as observing instrument. To find the necessary condition consider the secondary circuit AC_2EA. Since there is no current between A and E we have

Condition that Method may be "Null."

$$L\ddot{x} + M\ddot{u} + R_2\dot{x} = 0.$$

But if $\dot{z}$ be the current passing the condenser at this instant we must have (Fig. 127)

$$\dot{u} + \dot{x} = \dot{z},$$

and so $-\dot{x}$ is the current which charges the condenser. This gives

$$\ddot{x} = \ddot{z} - \ddot{u},$$

so that the former equation becomes

$$(M - L)\ddot{u} + L\ddot{z} + R_2\dot{x} = 0,$$

or

$$\dot{x} = -\frac{1}{R_2}\{(M - L)\ddot{u} + L\ddot{z}\}.$$

The charge of the condenser is then $CR_1\dot{z}$, so that

$$CR_1\dot{z} = -\int_0^t \dot{x}dt = \frac{1}{R_2}\{(M - L)\dot{u} + L\dot{z}\},$$

or

$$(M - L)\dot{u} = (CR_1R_2 - L)\dot{z}.$$

But in any case in which there has been ro integral flow through the galvanometer during the rising of the current from zero to its steady value we have seen that $CR_1R_2 = M$. Thus the equation just found becomes

$$(M - L)(\dot{u} - \dot{z}) = 0,$$

which asserts that either $M = L$, or $\dot{u} = \dot{z}$. The latter is only true when the current $\dot{u}$ in the battery has attained its steady value γ. If however $M = L$ it will be possible to make the difference of potential between A and E always zero and to employ a telephone.

Practical Example of Method.

The following results obtained in Prof. Carey Foster's laboratory by Mr. F. Womack illustrate the method. A small induction coil was used with fixed primary and coaxial secondary capable of being moved in the direction of the axis so as to alter the mutual inductance of the coils. The dimensions etc. of the coils were:—*Primary*, length 11·5 cms., mean radius 2 cms., wire 1·65 ohms of No. 20 B.W.G. *Secondary*, length 10·4 cms., inside radius 2·55 cms., outside radius 3·53 cms., wire 194 ohms of No. 30 B.W.G. Two Grove's cells were used and a condenser of 4·926 microfarads capacity, with a galvanometer of about 135 ohms resistance.

Experimental Results.

R_1	R_2 *Res. of Secondary + Res. from Box.*	$R_1 R_2 = M/C.$
15 ohms.	411 ohms.	6165×10^{16}
14 "	441 "	6174
13 "	476 "	6188
12 "	516 "	6192
11 "	561 "	6171
10 "	617 "	6170
9 "	684 "	6156
8 "	770 "	6160
7 "	882 "	6174
6 "	1029 "	6174
		Mean $6172·4 \times 10^{16}$

Thus in C.G.S. units

$$M = 4·926 \times 10^{-15} \times 6172 \times 10^{16} = 3·0403 \times 10^{7}.$$

The total resistance in the battery circuit was about $1·65 + ·6 + R_1$, or $R'_1 = 2·25$. Thus for greatest sensibility $R_2/R_1 = G/R'_1 = 135/2·25 = 60$.

Some very concordant results were also obtained with a 7 inch spark induction coil. The resistance of the primary was ·278 ohms; of the secondary 7394 ohms. One Grove's cell was used with the same condenser as before and a galvanometer of resistance 135·6 ohms.

R_1	R_2	$R_1 R_2$
27 ohms.	8944 ohms.	$2{\cdot}415 \times 10^{23}$ C.G.S.
28 ,,	8640 ,,	2·419
29 ,,	8334 ,,	2·417
30 ,,	8044 ,,	2·413
31 ,,	7784 ,,	2·413
32 ,,	7544 ,,	2·414
		Mean $2{\cdot}415 \times 10^{23}$ C.G.S.

Thus

$$M = 4{\cdot}926 \times 10^{-15} \times 2{\cdot}415 \times 10^{23} = 1{\cdot}1896 \times 10^{9},$$

in C.G.S. units.

CHAPTER IX

UNITS AND DIMENSIONS

In Volume I., Chapter III., a short account is given of the Theory of Dimensions, with a discussion of Fundamental and Derived Units as far as ordinary dynamical and electrostatic quantities are concerned. In the present chapter the subject of electric units is dealt with from a somewhat different point of view, and we therefore begin with electrostatic units, repeating, with modifications, a few paragraphs from the former chapter, in order that the discussion of electric and magnetic units here given may from that point of view be complete. For distinction here we shall, as a rule, use in the case of those quantities which appear in both the electrostatic and electromagnetic systems of units small letters for quantities taken in electrostatic measure, and the corresponding capitals of these letters for the same quantities taken in electromagnetic measure.

DERIVED ELECTRICAL UNITS.

I. Electrostatic System.

Quantity of Electricity.

Quantity of Electricity [q]. In the electrostatic system of units which is convenient when electrostatic results, independently of their bearing on electromag-

netic phenomena, are required, the units of all the other quantities are founded on the following definition of unit quantity of electricity. *Unit quantity of electricity is that quantity which, concentrated at a point at unit distance from an equal and similar quantity, also concentrated at a point, is repelled with unit force when the medium across which the electric action is transmitted is a certain standard insulating medium.* An ideal vacuum is sometimes taken as standard, but we shall suppose at present that the medium is air at temp. 0° C. and at standard atmospheric pressure. We shall call this simply *air*.

This definition is precisely similar to the definition (p. 516 above) of unit magnetic pole which forms the basis of another system of units called the *electromagnetic system*, of much wider and more important application than the electrostatic. Hence by Coulomb's law that (the numerical values of) electric attractions and repulsions are directly as the products of the (numerics for the) attracting and repelling quantities, and inversely as the second power of the (numeric for the) distance between them, if a quantity of positive electricity expressed by q be placed at a point distant L units from an equal quantity of electricity, then the medium being air, the numeric F for the force between them is q^2/L^2.

If the medium across which the electric action is transmitted be some other medium than air, the force between the charges is numerically q^2/KL^2 where K is the numerical measure of a quantity called the *electric inductive capacity*, or usually the *specific inductive capa-*

city of the medium. This quantity is precisely analogous to the conductivity of a substance for heat* and to magnetic permeability (see p. 517 below). In the ordinary electrostatic system of units it is defined (as at p. 514) so as to have a dimensional formula 1, that is, to be a mere numeric.

Dimensions of Specific Inductive Capacity.

But we might proceed otherwise and regard K as a quantity of undetermined dimensions as regards the fundamental units, but such that q^2/KL^2 has the dimensions of a force. We may then, in the absence of special reasons for preferring one dimensional formula for K to another, assign its dimensions according to any convenient hypothesis. One such hypothesis is that which forms the basis of the ordinary electrostatic system, namely, that K is, as regards the fundamental units, of zero dimensions, that is, has a dimensional formula [1]. But in the ordinary electromagnetic system of units, which has quite a different derivation from the electrostatic, the dimensional formula of K is $[L^{-2}T^2]$, and the numerical value of K depends on the choice made of fundamental units.

We shall in what follows suppose the dimensions of K undetermined, and therefore allow the symbol K expressing it to appear in the dimensional formulas of the other quantities. We shall thus obtain a more general electrostatic system in which the absolute dimensions of the quantities are not settled. From this the ordinary electrostatic system is obtained by simply deleting K.†

* See Vol. I. Chap. I. Sect. V.

† This method of proceeding is advocated, and its advantages pointed out, by Prof. A. W. Rücker, F.R.S., in a paper on the "Suppressed Dimensions of Physical Quantities," *Phil. Mag.*, Feb. 1889.

The dimensional formula of quantity of electricity is accordingly $[F^{\frac{1}{2}} L K^{\frac{1}{2}}]$ or $[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} K^{\frac{1}{2}}]$.

Electric Surface Density.

Electric Surface Density $[\sigma]$. The density of an electric charge on a surface is measured by the quantity of electricity per unit of area. Therefore $[\sigma]$ is $[qL^{-2}]$ or $[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} K^{\frac{1}{2}}]$.

Electric Force.

Electric Force and Intensity of Electric Field $[f]$. The electric force at any point in an electric field, or the intensity of the field at that point, is the force with which a unit of positive electricity would be acted on if placed at the point. Hence if the numeric for the quantity of electricity at a point P be q, and that of the electric force at that point be f, the numeric F for the force on the electricity is $q\,f$, and we have the equation $f = Fq^{-1}$. Therefore $[f]$ is $[Fq^{-1}]$ or $[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} K^{-\frac{1}{2}}]$.

Electric Potential.

Electric Potential $[v]$. The difference of electric potential between two points is measured by the work which would be done if a unit of positive electricity were placed at the point of higher potential and made to pass by electric force to the point of lower potential. Hence, in transferring q units of electricity through a difference of potential expressed numerically by v, an amount of work is done for which the numeric W is equal to qv. We have therefore $v = Wq^{-1}$, and hence $[v]$ is $[Wq^{-1}]$ or $[M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} K^{-\frac{1}{2}}]$.

Electrostatic Capacity.

Capacity of a Conductor $[c]$. The capacity of an insulated conductor is the quantity of electricity required to charge the conductor to unit potential, all other conductors in the field being supposed at zero potential. Hence, denoting the numeric for the capacity of a given conductor by c, those for its charge and

potential by q and v, we have $c = qv^{-1}$, and for $[c]$ therefore $[qv^{-1}]$ is $[LK]$. The unit of capacity has therefore the same dimensions as the unit of length provided $[K] = 1$; and the capacity of a conductor is then properly expressed as so many centimetres.

The electrostatic capacity of a conducting sphere is in ordinary electrostatic units numerically equal to the radius of the sphere. A conducting sphere of 1 cm. radius has therefore 1 C.G.S. unit of capacity.

Specific Inductive Capacity.

Specific Inductive Capacity $[K]$. The specific inductive capacity of a dielectric has already been virtually defined above, but it is usual to define it as the ratio of the capacity of a condenser, the space between the plates of which is filled with the dielectric, to the capacity of a precisely similar condenser with air as dielectric; or, according to Maxwell's * Theory of Electric Displacement, it is defined as the ratio of the electric displacement produced in the dielectric to the electric displacement produced in air by the same electric force. Thus in the ordinary electrostatic system of units its dimensions are taken as zero, that is, it is simply a numerical coefficient which does not change with the units. Hence in the ordinary electrostatic system $[K] = 1$.

This definition is quite consistent with the former as by assigning to each medium according to the former definition its own value of K, the capacities of condensers in which they are used as dielectrics have the ratios to one another given by the second. Thus if K

* *El. and Mag.*, Vol. I., 2nd edition, p. 154.

be taken as the expression of a physical property of the medium, which when fully known would in a natural manner fix the dimensions of K, it will be only necessary to multiply all the values of K obtained on an arbitrary supposition (such as, for example, that the value for air is unity) by the same factor depending on the units adopted. This, in fact, is what is done when K is taken in ordinary electromagnetic units.

Electric Current.

Electric Current $[\gamma]$. An electric current in a conducting wire is measured by the quantity which passes across a given cross-section per unit of time. If q be the numeric for the quantity which has passed in a time for which the numeric is T, then denoting the numeric for the current by γ, we have $\gamma = q/T$, and $[\gamma]$ is $[qT^{-1}]$ or $[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} K^{\frac{1}{2}}]$.

Resistance.

Resistance $[r]$. By Ohm's law the resistance of a conductor is expressed by the ratio of the numeric v for the difference of potential between its extremities to the numeric γ for the current flowing through it. We have therefore $r = v/\gamma$, and $[r]$ is $[v\gamma^{-1}]$ or $[L^{-1} TK^{-1}]$.

Conductance.

Conductance (formerly *Conductivity*). The dimensional formula of conductance is plainly $[LT^{-1} K]$. Hence in the ordinary electrostatic system its dimensional formula is $[LT^{-1}]$, which is that of velocity. Hence a conductance in ordinary electrostatic C.G.S. units is properly expressed in centimetres per second. A physical illustration of this fact, due to Sir William Thomson, is given in Vol. I. p. 205.

II.—Electromagnetic System.

Electromagnetic System of Units.

Magnetic Pole or Quantity of Magnetism $[m]$; *Surface Density of Magnetism* $[\sigma']$, *Magnetic Force or Magnetic Field Intensity* $[I]$; *Magnetic Potential* $[V]$.

The electromagnetic system of units is based on the unit magnetic pole as defined above (p. 2). This definition is exactly the same as that of unit quantity of electricity on which the electrostatic system is founded; and therefore the purely magnetic quantities here mentioned, which bear the same relations to the unit quantity of magnetism that the corresponding electric quantities bear to the chosen unit quantity of electricity, have, with the substitution of the magnetic analogue to K, in the electromagnetic system the same dimensional formulas as those just found for the latter quantities in the electrostatic system.

Observations precisely similar to those made above regarding specific inductive capacity apply here regarding its analogue, magnetic inductive capacity, or, as it is frequently called, magnetic permeability. The force between two poles, each of strength m at distance L, in a medium of magnetic inductive capacity μ, is numerically $m^2/\mu L^2$, and hence $[m] = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} \mu^{\frac{1}{2}}]$ In the ordinary electromagnetic system μ is defined (see p 517 below) so as to be a mere numeric We shall not here make this assumption, but allow μ to appear in the formulas, and its dimensions may be afterwards assigned.

By simple deletion of μ from the dimensional for-

mulas they become those for the ordinary electromagnetic system in which $[\mu] = 1$.

Magnetic Moment $[M]$. The numeric M, for the magnetic moment of a uniformly magnetized bar-magnet, is the product of the numerics for the strength of either pole and the length of the magnet. Hence we have $[M] = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} \mu^{\frac{1}{2}}] . [L] = [M^{\frac{1}{2}} L^{\frac{5}{2}} T^{-1} \mu^{\frac{1}{2}}]$. Magnetic Moment.

Intensity of Magnetization $[\nu]$. The intensity of magnetization of any portion of a magnet is measured by the magnetic moment of that portion per unit of volume. Hence, if ν denote the numeric for the intensity of magnetization of a uniformly magnetized magnet, the numerics for the magnetic moment and volume of which are M and AL^3, we have Intensity of Magnetization.

$$\nu = \frac{M}{AL^3}, \text{ and } [\nu] = [M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} \mu^{\frac{1}{2}}].$$

It is plain that the intensity of magnetization of a uniformly and longitudinally magnetized bar is equal to the surface density of the magnetic distribution over the ends of the bar, and therefore intensity of magnetization has the same dimensional formula as magnetic surface density.

Magnetic Permeability $[\mu]$. The magnetic permeability of an inductively magnetized substance, or its magnetic inductive capacity, is, as has already been stated, the analogue in magnetism of specific inductive capacity of a dielectric in electricity, and of the conductivity of a body for heat in heat conduction. The part which it plays in magnetic theory is discussed in Chapter I. above. In the ordinary electromagnetic Magnetic Permeability.

Magnetic Permeability and Susceptibility.

system of units it is usually defined as the ratio of the magnetic force which would be exerted on a unit pole placed in a narrow crevasse, cut in the substance so that its walls are at right angles to the direction of magnetization, to the force which it would experience if placed in a narrow crevasse, the walls of which are parallel to the direction of magnetization. This mode of defining magnetic permeability clearly makes it in the ordinary magnetic system a mere numeric, that is, its dimensional formula $[\mu] = 1$.

The more general view of the meaning of magnetic permeability given on p. 516 is not inconsistent with this more special definition, as the latter simply amounts to assuming the permeability of air to be unity. If, as may be the case, permeability is more properly measured by some property of the medium, which will assign to the quantity definite dimensions, the permeabilities of different substances in the ordinary electro-magnetic system will simply have to be multiplied by a common factor, depending on the fundamental units adopted.

Magnetic Susceptibility. This quantity is usually denoted by k, and in the ordinary electromagnetic system is connected with μ by the relation $\mu = 1 + 4\pi k$. Its dimensional formula is therefore also 1 in the ordinary electromagnetic system, that is, magnetic susceptibility is in that system a mere numeric.

Current Strength $[\Gamma]$ By the theory of electromagnetic action stated above in p. 143, and the definition of unit current (3), p. 144, we have, for any actual case of a magnetic pole placed at the centre of a circle of wire carrying a current, the equation $\Gamma = FL/2\pi m$,

Dimensions of Electrical Quantities.

where F, L, m, and Γ are the numerics respectively for the force acting on the pole, the radius of the circle, the strength of the pole, and the strength of the current. Hence $[\Gamma] = [FLm^{-1}] = [M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} \mu^{-\frac{1}{2}}]$.

Quantity of Electricity $[Q]$. The numeric Q for the quantity of electricity conveyed in T seconds by a current the numeric for the strength of which is Γ, is equal to ΓT. Hence $[Q] = [\Gamma T] = [M^{\frac{1}{2}} L^{\frac{1}{2}} \mu^{-\frac{1}{2}}]$.*

Electric Potential, or Electromotive Force $[V]$. As above (p. 513), but using in this case the symbol V for a numerical difference of potential, we get $W = VQ$. Thus we have $[V] = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} \mu^{\frac{1}{2}}]$.

Electrostatic Capacity $[C]$. Using for a numerical capacity in electromagnetic units the symbol C, we find, by the same process as in p. 513, the equation $C = Q/V$, $[C] = [L^{-1} T^2 \mu^{-1}]$.

Illustration of Resistance as measured by a Velocity.

Resistance $[R]$. Using here R to denote a numerical resistance, we get as formerly $R = V/C$, and therefore $[R] = [LT^{-1} \mu]$.

Thus if $[\mu] = 1$, the dimensional formula for resistance is the same as that for velocity, and therefore a resistance in ordinary electromagnetic units is properly expressed as a velocity is, in units of length per unit of time, and accordingly, in C.G.S. units, as so many centimetres per second. This fact is directly shown by the

* We might pass in the electrostatic system from the dimensional formula of unit current to that of unit quantity of magnetism, precisely as we pass here in the electromagnetic system from the dimensional formula of unit quantity of magnetism to that of unit current, and we should find for the dimensional formula sought that here obtained $[M^{\frac{1}{2}} L^{\frac{1}{2}} K^{-\frac{1}{2}}]$, as might be inferred at once. From this the formulas in the electrostatic system for all the other magnetic quantities might be found.

following illustration, due to Sir William Thomson. Let the rails of the ideal machine, described in p. 192, be supposed to run horizontally at right angles to the magnetic meridian, and let their plane be vertical. Let a tangent galvanometer be included in the wire connecting the rails. The slider when moved along the rails will cut the lines of the earth's horizontal force, the intensity of which in electromagnetic measure we have denoted by H. If the slider have a length L, and be moved with a velocity v, the electromotive force developed will be HLv. If R be the total resistance in circuit, γ the current flowing, r the mean radius of the galvanometer coil, and L' the length of wire in the coil, we have $\gamma = Hr^2/L' \tan\theta$. But by Ohm's law $\gamma = HLv/R$. Hence $HLv/R = Hr^2/L' \tan\theta$, or

$$R = \frac{LL'v}{r^2 \tan\theta}.$$

Now we may suppose the radius r of the coil so taken that $r^2 = LL'$, and that the slider is moved at such a speed, v, that the deflection of the needle is 45°. Under these conditions we get $R = v$. The resistance R of the circuit is therefore measured in electromagnetic units by the velocity with which the slider must be moved, so that the deflection of the needle of the tangent galvanometer may be 45°.

Self-Inductance.

Coefficient of Self-Induction (or Self-Inductance). Denoting by L (instead of L to avoid confusion with the L of the unit of length) the inductance of a circuit, the current in which is Γ, we have $\mathsf{L}d\Gamma/dt$ for the electromotive force of self-induction. Hence $\mathsf{L}d\Gamma/dt$

has the same dimensional formula as electromotive force, that is $[\mathfrak{L}\Gamma T^{-1}] = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} \mu^{\frac{1}{2}}]$, and therefore $[\mathfrak{L}] = [L\mu]$.

Mutual Inductance. If M be mutual inductance between two circuits, Γ the current in one of them, then the electromotive force in the other circuit due to mutual induction is $Md\Gamma/dt$. Hence by the same process as before we get $[M] = [L\mu]$. Mutual Inductance.

A self- or mutual inductance is therefore in ordinary electromagnetic measure in dimensions simply a length, and in C.G.S. units is properly expressed as so many centimetres.

But if resistance is taken in terms of the true ohm, which is 10^9 cms. (or nearly one earth-quadrant) per second, the corresponding unit of induction is 10^9 cms. If the legal or any other ohm is used, the unit of induction is that length which replaces 10^9 cms. in the definition of the ohm.

For the unit of inductance defined by *any* ohm, Profs. Ayrton and Perry have proposed the name *secohm*. The Paris Congress has however adopted the name *quadrant*. Plainly this can only in strictness be applied in connection with the true ohm.

We have now investigated the dimensional formulas of the absolute units of all the principal electric and magnetic quantities in the electrostatic system, or in the electromagnetic system, according as each quantity is generally measured in practice. Each may, however, be expressed either in electrostatic or in electromagnetic units, and we give the following table of dimensional formulas for all the quantities in both systems.

In Tables II. and III. K and μ have been introduced into the formulas as stated above, pp. 512, 516. The ordinary electrostatic and electromagnetic systems are obtained by supposing K and μ each unity.

Suppressed Dimensions of K and μ.

One advantage of thus exhibiting the dimensions is that it enables electrostatic and electromagnetic quantities to be regarded as of the same absolute dimensions, since K and μ, not being fixed as to dimensions, can, unless restricted by definition, have dimensions assigned to them which fulfil this condition. For example, as suggested by Professor G. F. Fitzgerald,* each may be taken as having the dimensions $[TL^{-1}]$. Another advantage is that problems, in which passage from one set of units to the other is involved, are solved with greater ease from first principles (see Professor Rucker's paper, *loc. cit.*).

Table of Dimensional Formulas.

FUNDAMENTAL UNITS.

Quantity.	Dimensional Formula.
Length	$[L]$
Mass	$[M]$
Time	$[T]$

DERIVED UNITS.

I. Dynamical Units.

Velocity	$[L T^{-1}]$
Acceleration	$[L T^{-2}]$
Force	$[M L T^{-2}]$
Work / Energy	$[M L^2 T^{-2}]$

* *Phil. Mag.*, April 1889.

NOTE.—Col. A below with K deleted gives the ordinary electrostatic formulas, Col. B with μ deleted gives the ordinary electromagnetic formulas.

II. Electric Units.

	A. In terms of L, M, T, K.	B. In terms of L, M, T, μ.
Quantity of Electricity	$[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} K^{\frac{1}{2}}]$	$[M^{\frac{1}{2}} L^{\frac{1}{2}} \mu^{-\frac{1}{2}}]$
Surface Density of Electricity, Electric Displacement	$[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} K^{\frac{1}{2}}]$	$[M^{\frac{1}{2}} L^{-\frac{3}{2}} \mu^{-\frac{1}{2}}]$
Electric Force, or Intensity of Electric Field	$[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} K^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-2} \mu^{\frac{1}{2}}]$
Electric Potential, Electromotive Force	$[M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} K^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} \mu^{\frac{1}{2}}]$
Electrostatic Capacity	$[K]$	$[L^{-2} T^2 \mu^{-1}]$
Specific Inductive Capacity	$[L K]$	$[L^{-1} T \mu^{-1}]$
Current Strength	$[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} K^{\frac{1}{2}}]$	$[M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} \mu^{-\frac{1}{2}}]$
Resistance	$[L^{-1} T K^{\frac{1}{2}}]$	$[L T^{-1} \mu^{-\frac{1}{2}}]$

III. Magnetic Units.

	A.	B.
Quantity of Magnetism, or Magnetic Pole	$[M^{\frac{1}{2}} L^{\frac{1}{2}} K^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} \mu^{\frac{1}{2}}]$
Surface Density of Magnetism	$[M^{\frac{1}{2}} L^{-\frac{3}{2}} K^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} \mu^{\frac{1}{2}}]$
Magnetic Moment	$[M^{\frac{1}{2}} L^{\frac{3}{2}} K^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}} L^{\frac{5}{2}} T^{-1} \mu^{\frac{1}{2}}]$
Intensity of Magnetization	$[M^{\frac{1}{2}} L^{-\frac{3}{2}} K^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} \mu^{\frac{1}{2}}]$
Magnetic Force or Intensity of Magnetic Field	$[M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-2} K^{\frac{1}{2}}]$	$[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} \mu^{-\frac{1}{2}}]$
Magnetic Potential	$[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} K^{\frac{1}{2}}]$	$[M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} \mu^{-\frac{1}{2}}]$
Magnetic Permeability, Magnetic Susceptibility	$[L^{-2} T^2 K^{-1}]$	$[\mu]$
Coefficient of Self-Induction, Coefficient of Mutual Induction	$[L^{-1} T^2 K^{-1}]$	$[L \mu]$

Use of Dimensional Formulas.

As an example of the use of dimensional formulas we may find the multiplier for the reduction of numerics for magnetic field intensities given in terms of British foot-grain-second units to the corresponding numerics in terms of C.G.S. electromagnetic units. Let H be the numerical intensity in terms of British units, H' the numerical intensity in C.G.S. units. We have, by equation (4).

$$H' = H[H] = H[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}].$$

Since 1 gramme = 15·43235 grains, and 1 centimetre = 1/30·47945 foot, we have

$$[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}] = \left(\frac{1}{15{\cdot}43235 \times 30{\cdot}47945}\right)^{\frac{1}{2}} = \frac{1}{21{\cdot}688}.$$

The earth's horizontal force is given as 3·92 in British units at Greenwich for 1883. We get therefore

$$H' = 3{\cdot}92 \frac{1}{21{\cdot}688} = {\cdot}18075, \text{ in C.G.S. units.}$$

Units adopted in Practice.

Practical Units.

In practical work the resistances and electromotive forces occurring to be measured are usually so great that if the absolute electromagnetic C.G.S. units were used, the resulting numerics would be inconveniently large; while, on the other hand, capacities are generally so small that their numerics in C.G.S. units would be only very small fractions. Accordingly certain multiples of the C.G.S. units of resistance and electromotive force, and a submultiple of that of capacity have been chosen for use in practice. The derivation of the first two, the

ohm and the volt, together with the practical units of current and quantity, the ampere and the coulomb, may be illustrated as follows:—*

Derivation of Practical Units.

Let the rails of the ideal magneto-electric machine, described at p. 192 above, be imagined placed in a uniform magnetic field of unit intensity. Also let the rails be connected by means of a wire so that a complete conducting circuit is formed. Suppose the rails, slider, and wire to be all made of the same material, and the length and cross-sectional area of the wire to be such that its resistance is very great in comparison with that of the rest of the circuit, so that, when the slider is moved with any given velocity, the resistance in the circuit remains practically constant. When the slider is moved along the rails it cuts across the lines of force, and so long as it moves with uniform velocity a constant difference of potential will be maintained between its two ends by induction, and a uniform current will flow in the wire from the rail which is at the higher potential to that which is at the lower. If the direction of the lines of force be the same as the direction of the vertical component of the earth's magnetic force in the northern hemisphere, so that a north-tending pole placed in the field would be moved downwards, and if the rails run south and north, the current, when the slider is moved northwards, will flow from the east rail to the west through the slider, and from the west rail to the east

* Specifications of the Practical Units for use in Electrical Industries, &c., have been adopted within the last year by a Committee appointed by the Board of Trade. The recommendations of the Committee will be found in an Appendix at the end of this volume.

through the wire. If the velocity of the slider be increased, the difference of potential between the rails, or, as it is otherwise called, the electromotive force producing the current, will be increased in the same ratio; and therefore by Ohm's law so also will the current.

C.G.S. unit of Difference of Potential.

For a slider arranged as we have imagined, and made to move across the lines of force of a magnetic field, the difference of potential produced varies directly as the field intensity, as the length of the slider, and as the velocity with which the slider cuts across the lines of force. The difference of potential produced therefore varies as the product of these three quantities; and when each of these is unity, the difference of potential is taken as unity also. We may write therefore $V = I L v$, where I is the field intensity, L the length of the slider, and v its velocity. Hence if the intensity of the field we have imagined be 1 C.G.S. unit, the distance between the rails 1 cm., and the velocity of the slider 1 cm. per second, the difference of potential produced will be 1 C.G.S. unit.

Practical Unit of Electromotive Force: The Volt.

This difference of potential is so small as to be inconvenient for use as a practical unit, and instead of it the difference of potential which would be produced if, everything else remaining the same, the slider had a velocity of 100,000,000 cms. per second, is taken as the practical unit of electromotive force, and is called one *volt*. It is a little less than the difference of potential which exists between the two insulated poles of a Daniell's cell.

We have imagined the rails to be connected by a wire

of very great resistance in comparison with that of the rest of the circuit, and have supposed the length of this wire to have remained constant. But from what we have seen above, the effect of increasing the length of the wire, the speed of the slider remaining the same, would be to diminish the current in the ratio in which the resistance is increased, and a correspondingly greater speed of the slider would be necessary to maintain the current at the same strength. We may therefore take the speed of the slider as measuring the resistance of the wire. Now suppose that when the slider 1 cm. long was moving at the rate of 1 cm. per second, the current in the wire was 1 C.G.S. unit; the resistance of the wire was then 1 C.G.S. unit of resistance. Unit resistance therefore corresponds to a velocity of 1 cm. per second. This resistance, however, is too small to be practically useful, and a resistance 1,000,000,000 times as great, that is, the resistance of a wire, to maintain 1 C.G.S. unit of current in which it would be necessary that the slider should move with a velocity of 1,000,000,000 ms. (approximately the length of a quadrant of the earth from the equator to either pole) per second, is taken as the practical unit of resistance, and called one *ohm*.

Derivation of C.G.S. Unit of Resistance.

Practical Unit of Resistance: The Ohm.

Some account of experiments which have been made for the realization of the ohm is given in Chapter IX., and it will be seen from the results that the ohm is approximately equal to the resistance of a column of pure mercury 106.3 cms. long, and 1 sq. mm. in cross-section, when the temperature is that of melting ice. At present this number is that generally used; but

Ohm realized as Resistance of Column of Mercury

when a sufficient number of exact determinations of the ohm have been obtained, a more accurate value will no doubt be legalised as the standard of reference.

Derivation of the Ampere.

It is obvious from equation (1) that if V and R, each initially one unit, be increased in the same ratio, C will remain one unit of current; but that if V be, for example, 10^8 C.G.S. units of potential, or one volt, and R be a resistance of 10^9 cms. per second, or one ohm, C will be one-tenth of one C.G.S. unit of current. A current of this strength—that is, the current flowing in a wire of resistance one ohm, between the two ends of which a difference of potential of one volt is maintained,—has been adopted as the practical unit of current, and called one *ampere*. Hence it is to be remembered one ampere is one tenth of one C.G.S. unit of current.

Derivation of the Coulomb.

The amount of electricity conveyed in one second by a current of one ampere is called one *coulomb*. This unit, although not quite so frequently required as the others, is very useful, as, for instance, for expressing the quantities of electricity which a secondary cell is capable of yielding in various circumstances. For example, in comparing different cells with one another, their capacities, or the total quantities of electricity they are capable of yielding when fully charged, are very conveniently reckoned in coulombs per square centimetre of the area across which the electrolytic action in each takes place.

The magneto-electric machine we have imagined yields very simply the relation between the work done in maintaining a current, the strength of the current,

and the electromotive force producing it.* Although a slight digression, we give this discussion here to help to illustrate how electromotive force and current together, when measured in absolute electromagnetic units, give the corresponding electrical activity in absolute dynamical units, and for the sake of the practical units of electrical activity and work.

Activity in Circuit of Electromagnetic Generator.

We have seen above (p. 118) that every element of a conductor, carrying a current in a magnetic field, is acted on by a force tending to move it in a direction at right angles to the plane through the element, and the direction of the resultant magnetic force at the element, and have derived from the expression for the magnitude of the force a definition (p. 143) of unit current in the electromagnetic system. From these considerations it follows that a conductor in a uniform magnetic field, and carrying a unit current which flows at right angles to the lines of force, is acted on at every point by a force tending to move it in a direction at right angles to its length, and the magnitude of this force for unit length of conductor, and unit field, is by the definition of unit current equal to unity.

Applying this to our slider, in which we may suppose a current of amount γ to be kept flowing, say, from a battery in the circuit, let L be the length of the slider, v its velocity, and I the intensity of the field; we have

* Practically the same considerations formed the basis of Sir William Thomson's famous papers "On the Mechanical Theory of Electrolysis," and "On Applications of the Principle of Mechanical Effect to the Measurement of Electromotive Forces and of Galvanic Resistances in Absolute Measure," *Phil. Mag.* Dec. 1851, or *Reprint of Math. and Phys. Papers*, Vol. I.

for the force on the moving conductor the value $IL\gamma$. Hence the rate at which work is done by the electromagnetic action between the current and the field is $IL\gamma dv/dt$ or $IL\gamma v$, and this must be equal to the rate at which work would be done in generating by motion of the slider a current of amount γ. But as we have seen above, ILv is the electromotive force produced by the motion of the slider. Calling this now E, the symbol usually employed to denote electromotive force, we have $E\gamma$ as the electrical activity, that is, the total rate at which electrical energy is given out in all forms in the circuit.

By Ohm's law this value for the electrical activity may, when the work done is wholly spent in producing heat, be put into either of the two other forms, namely, E^2/R, or $\gamma^2 R$. In the latter of these forms the law was discovered by Joule, who measured the amount of heat generated in wires of different resistances by currents flowing through them. This law holds for every electric circuit whether of dynamo, battery, or thermoelectric arrangement.

Activities in Different Parts of Circuit. We have, in what has gone before, supposed the slider to have no resistance comparable with the whole resistance in the circuit. If it have a resistance r, and R be the remainder of the resistance in circuit, the actual difference of potential between its two ends will not be ILv or E, but $E \cdot R/(R+r)$ (Vol. I., p 146). The rate per unit of time at which work is given out in the circuit is however still $E\gamma$, of which the part $E\gamma \cdot r/(R+r)$ is given out in the slider, and the remainder, $E\gamma \cdot R/(R+r)$, in the remainder of the circuit. In

short, if V be the actual difference of potential, as measured by an electrometer, between two points in a wire connecting the terminals of a battery or dynamo, and γ be the current flowing in the wire, the rate at which energy is given out is $V\gamma$, or if R be the resistance of the wire between the two points, $\gamma^2 R$.

The activity in the part of the circuit considered is always $V\gamma$, but this may be greater than $\gamma^2 R$, in which case work is done otherwise than in heating the conductor. $\gamma^2 R$ is then the part of the activity employed in generating heat.

One of the great advantages of the system of units briefly sketched here, is that it states the value of the rate at which work is given out in the circuit, without its being necessary to introduce any coefficient such as would have been necessary if the units had been arbitrarily chosen. When the quantities are measured in C.G.S. units, the value of $E\gamma$ is given in centimetre-dynes, or in *ergs*, per second. Results thus expressed may be reduced to *horse-power* by dividing by the number $7{\cdot}46 \times 10^9$; or if E is measured in volts, and γ in amperes, $E\gamma$ may be reduced to horse-power by dividing by 746. Thus, if 90 volts be maintained between the terminals of a pair of incandescent lamps joined in series, and a current of 1·3 ampere flows through these lamps, the rate at which energy is given out in the lamps is approximately ·157 horse-power. If the rate at which work is done in maintaining a current of one ampere through a resistance of one ohm were taken as the practical unit of rate of working, or *activity*, and E reckoned in volts and γ in amperes, the

Advantage of C.G.S. System of Units.

rate at which electrical energy is given out in the circuit would be measured simply by $E\gamma$; and calculations of electrical work would be much simplified. This was proposed by Sir William Siemens (Brit. Assoc. Address 1882), who suggested that the name *watt* should be given to this unit rate of working. The rate at which energy is given out in the lamps of the above example is $90 \times 1{\cdot}3 = 117$ *watts*. A *watt* is therefore equivalent to 10^7 ergs per second.

Watt or Unit of Electrical Activity.

Kilowatt.

The Electrical Congress held at Paris in August 1889 has adopted the watt as the practical electrical unit of work, and the term *kilowatt*, proposed by Mr. W. H. Preece to designate an activity of 1000 watts or 10^{10} ergs per second. It is intended that the latter unit should be used instead of the horse-power. An activity given in kilowatts can be reduced to horse-power by dividing by 746, or roughly by multiplying by 4 and dividing by 3.

Joule or Unit of Electrical Work.

Sir William Siemens also proposed to call the work done in one second, when the rate of working is one watt, one *joule*. Thus the work done in one second in maintaining a current of one ampere through one ohm, or the work obtained by letting down one coulomb of electricity through a difference of potential of one volt, is one *joule*. A *joule* is therefore equivalent to 10^7 ergs, and the work done in one second in the above example is 117 *joules*.

The Electrical Congress of August 1889 also adopted the joule as the practical unit of electrical work.

The practical unit of electrostatic capacity is called the *farad*, and may be defined as the capacity of a

condenser which, when charged by an electromotive force of one volt, has a charge of one coulomb. If C be the numerical capacity of such a condenser in C.G.S. electromagnetic units of capacity, we have $C = 10^{-1}/10^8 = 10^{-9}$; or one farad is equivalent to 10^{-9} C.G.S. Practical Unit of Capacity: The Farad

In some cases, when the quantities to be expressed are very large, units one million times greater than the chosen practical units are employed. These are denoted by the names of the corresponding practical units with *mega* (great) prefixed. On the other hand, for the expression of very small quantities, units one million times smaller than the practical units are sometimes used, and are denoted by the corresponding names of the practical units with *micro* (small) prefixed.

Such units are however rarely employed, with the exception of the *megohm*, used for expressions of resistances of insulating substances, and the *microfarad*, which is really the most convenient unit for expressions of capacities. A megohm may be stated as 10^{15} centimetres per second; one C.G.S. unit of capacity is equivalent to 10^{15} microfarads. Megohm and Microfarad.

The practical units which have been adopted may be considered as belonging to an absolute system, based on a unit of length equivalent to one thousand million (10^9) centimetres (approximately the length of one quadrant of the earth's polar circumference), a unit of mass equivalent to one one-hundred-millionth of a milligramme, or 10^{-11} gramme, and the second as unit of time. The verification of this in the different cases will furnish examples of the use of dimensional formulas. Practical Units regarded as Absolute System.

For example, let us find what the expressions of resistances and electromotive forces in C.G.S. units become when these new units of length and mass are substituted for the centimetre and the gramme. Let R be the numeric of a resistance in C.G.S. units, and R' its numeric in terms of the new units. We have, using the proper change-ratio,

$$R' = R[LT^{-1}] = R \times \frac{1}{10^9},$$

since the unit of time remains unchanged. One ohm is therefore equivalent to 10^9 C.G.S. units of resistance, that is, 10^9 centimetres per second.

Again let V be the expression of an electromotive force in C.G.S. electromagnetic units, E' its expression in terms of the new units. The dimensional formula for electromotive force is $[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}]$. We have therefore

$$E' = E[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}].$$

We have only to consider what $[M^{\frac{1}{2}} L^{\frac{3}{2}}]$ becomes. This is plainly $(1/10^{-11})^{\frac{1}{2}} \times (1/10^9)^{\frac{3}{2}}$ or $1/10^8$. Hence

$$E' = E \times \frac{1}{10^8},$$

that is, one volt is equivalent to 10^8 C.G.S. units of electromotive force.

The following table gives the numerics for the various practical units in terms of C.G.S. units:—

Name of Quantity.		Practical Unit.	Equivalent in C.G.S. Units.
Resistance		Ohm	10^9
Electromotive Force		Volt	10^8
Current Strength		Ampere	10^{-1}
Quantity of Electricity		Coulomb	10^{-1}
Electrostatic Capacity	{	Farad	10^{-9}
		Microfarad	10^{-15}

We have seen above that if Q, Q' be the numerics for two quantities, the dimensional formula of Q'/Q is $[Q']/[Q]$, and this of course applies to the expressions of the same quantity in two different systems of units. Thus if q denote the numerical expression of a quantity of electricity in electrostatic units, and Q that of the same quantity in electromagnetic units, the same fundamental units being employed in both cases, the dimensional formula of q/Q is $[q]/[Q]$. But from the table (p. 523) we have $[q] = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}]$ and $[Q] = [M^{\frac{1}{2}} L^{\frac{1}{2}}]$. The dimensional formula of q/Q is thus the same as that of velocity, that is to say q/Q is equal to a *certain definite* velocity, the numerical expression of which depends on the fundamental units of length and time employed. In other words the number of *electrostatic* units of electricity which is equivalent to one *electromagnetic* unit is numerically equal to this velocity.

Dimensions of Ratio of Electromagnetic to Electrostatic Unit of Quantity.

The same velocity is derivable from the ratios of the numerical values of any of the other electrical or magnetic quantities in the two systems of units. For instance, if e be the numerical value of an electromotive force in electrostatic units, and E that of the same electromotive force in electromagnetic units, we have

for the dimensional formula of e/E the value $[e]/[E] = [M^{\frac{1}{2}} L^{\frac{1}{2}}]/[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}] = [L^{-1} T]$. The ratio e/E has thus the dimensional formula of the reciprocal of a velocity, and therefore E/e, or which is the same, the number of electromagnetic units equivalent to one electrostatic unit of electromotive force, is properly expressed as a certain definite velocity. It is easy to see that this velocity is identical with the former. For if q and Q be the numerical values in the two systems of a certain quantity of electricity, then since e and E denote the same electromotive force, the work eq must be numerically equal to the work EQ. We get therefore $E/e = q/Q$, that is, the two velocities are the same.

By taking the more general dimensional formulas given in the table (p. 523) we find that

$$\frac{[q]}{[Q]} = [K^{-\frac{1}{2}} \mu^{-\frac{1}{2}}]$$

when $[q]$, $[Q]$ refer to the ordinary systems. Hence the product $K^{-\frac{1}{2}} \mu^{-\frac{1}{2}}$ has the dimensions of a velocity. It *is* in fact the velocity q/Q above referred to.*

Denoting this velocity by v,† we get for the various quantities the following relations. The numerator of the ratio on the left of each equation denotes the numeric of the quantity in electrostatic units, the denominator the numeric of the same quantity in electromagnetic units.

* See p. 206 above; also Chap. X. below.

† For illustrations of the physical meaning of v see Chap. X. below.

A given	Quantity of Electricity	$q \; Q = v$
,,	Current	$\gamma \; \Gamma = v$
,,	Electromotive Force	$e/E = 1/v$
,,	Electrostatic Capacity	$c \; C = v^2$
,,	Resistance	$r/R = 1/v^2$

Therefore if q and Q, e and E, or the numerics for any other given quantity, be determined in the two systems of units, the value of v can be at once obtained. Experiments of this kind, some of which are described in Chapter X. below, have been made by Maxwell, Sir W. Thomson, Weber, Ayrton and Perry, J. J. Thomson, H. A. Rowland, E. B. Rosa, and others, with the result that $v = 3 \times 10^{10}$ centimetres per second approximately, or very nearly the velocity of light in air as deduced from experiments made by the methods of Foucault and Fizeau. According to Maxwell's Electromagnetic Theory of Light (*Electricity and Magnetism*, Vol. II., Chap. XX.) this relation should hold, and thus the theory is so far confirmed.

In the present chapter we have considered only the scalar magnitudes of electric and magnetic quantities. For an instructive discussion of the Theory of Dimensions from a vector point of view the reader may refer to a paper by Mr. W. Williams, *Phil. Mag.* Sept. 1892.

CHAPTER X

ABSOLUTE MEASUREMENT OF RESISTANCE

Importance of Realized Standards of Resistance.

In order that all the results of electrical experiments may be expressed in absolute units, realized absolute units of resistance must be available. An electric current can be measured at any time in absolute units, as we have seen, by means of a proper standard galvanometer or current balance. When the absolute value, R, of the resistance of a coil of wire is known, a difference of potential expressed by any chosen number of absolute units can be produced by causing a current of the proper strength, γ, to flow through the wire. If the wire is not the seat of any electromotive force, the difference of potential between two points in the wire, close to the ends, is γR. By this mode of realizing differences of potential the electromotive forces of voltaic cells have been determined; and such cells can be used in their turn as practical standards for the comparison of differences of potential. A realized standard of resistance is thus of fundamental importance in absolute electrical measurement.

Absolute Measurement of Resistance.

Various methods for the absolute measurement of resistances have been devised, and a few of these most suited to give exact results have been carried out with great care and experimental skill by several experi-

menters. We give here a general account of these investigations, going however into full detail regarding only one or two of the more recent, and, on account of the accumulation of experience, presumably the more exact of them.

The methods may be classed in three divisions:— I. Those in which electromagnetic induction, of which the amount can be calculated, is employed to generate a current in the conductor the resistance of which is to be determined. The strength of this current depends on this resistance, and is measured directly or indirectly so that it enables the resistance to be found. II. Those based on Lorenz's method, in which a continuous difference of potential between the terminals of the given conductor is produced by electromagnetic induction, and is balanced by a difference of potential independently produced by a current γ flowing in the conductor. III. Joule's method, in which the rate, $\gamma^2 R$, of generation of heat produced by a measured current γ in the conductor is determined, and the resistance deduced by dividing by γ^2. Methods

The first method of type I. which we describe is that due to Kirchhoff.* Two coils, C_1, C_2, between which there is a mutual inductance, M, are joined up, as shown diagrammatically in Fig. 128, with a battery and galvanometer, and the resistance R to be determined. The steady current deflection of the needle is first observed. C_1 is then removed from the position in which the mutual inductance is M, to one in which the mutual inductance is zero, and the first throw of the galvano- Kirchhoff's Method.

* *Pogg. Ann.* 76, (1849).

meter is noted (together with the succeeding elongations to enable a correction for damping to be applied) If x be the total induction-flow through the galvanometer, i the steady current in G, and the resistances P, Q, of AC_1B, BC_2A, respectively, be each great in comparison with R, then we have very approximately

$$R = M\frac{i}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the galvanometer deflections for steady currents follow the tangent law, and θ_1 be the deflection produced by the steady current, θ_2 the induction throw, corrected

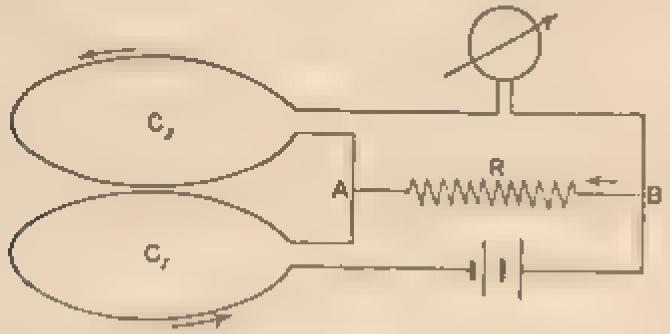


Fig. 128.

for damping and torsion of the fibre if it exists, and T the complete period of oscillation of the needle,

$$\frac{i}{x} = \frac{\pi}{T}\frac{\tan\theta_1}{\sin\frac{1}{2}\theta_2}$$

so that

$$R = M\frac{\pi}{T}\frac{\tan\theta_1}{\sin\frac{1}{2}\theta_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If two different galvanometers are used, one of constant G_1 to measure the steady current, and a ballistic galvanometer of constant G_2 to measure the transient current,

and H, H', be the values of the earth's horizontal components at their respective needles, then instead of (2) we have

$$R = M\frac{H}{H'}\frac{G_2}{G_1}\frac{\pi\tan\theta_1}{T\sin\frac{1}{2}\theta_2} \quad . \quad . \quad . \quad . \quad (2')$$

Theory of Kirchhoff's Method.

To prove equation (1) let $\dot u$, $\dot x$, be the current in the battery and in the galvanometer at any instant during the change of the inductance, and L_1, L_2, the self-inductances of the two circuits AC_1BA, ABC_2A, R being supposed devoid of self-inductance. Then these circuits give

$$\left.\begin{array}{l} L_1\ddot u + M\ddot x + (P+R)\dot u - R\dot x = E \\ L_2\ddot x + M\ddot u + (Q+R)\dot x - R\dot u = 0 \end{array}\right\} \quad . \quad . \quad (3)$$

where E is the electromotive force of the battery.

Integrating these equations over the (very short) interval τ of change of the mutual inductance from M to 0, we get

$$\left.\begin{array}{l} -M\dot x_s + (P+R)u - Rx - \int_0^\tau Edt = 0 \\ -M\dot u_s + (Q+R)x - Ru = 0 \end{array}\right\} . \quad . \quad . \quad (4)$$

But when the currents are steady the second of (3) is

$$(Q+R)\dot x_s - R\dot u_s = 0.$$

Eliminating u between the two equations of (4), and putting $\dot u_s = \dot x_s(Q+R)/R$, as given by the last equation, we find after reduction

$$\frac{x}{\dot x_s} = \frac{M}{R}\frac{(P+R)(Q+R)+R^2}{(P+R)(Q+R)-R^2}$$

$$= \frac{M}{R}\left\{1 + \frac{2R^2}{(P+R)(Q+R)} + \&c\right\} \quad . \quad . \quad . \quad (5)$$

If P, Q, be each great in comparison with R, this gives (1). Equation (2) follows by the theory of the ballistic galvanometer.

This investigation is practically Maxwell's version of the process followed originally by Kirchhoff. The result may however be obtained somewhat more directly as follows. When M is annulled an integral electromotive force of amount $M\dot u_s$ acts

in C_2, and another of amount $M\dot{x}_s$ in C_1. The induction-flow due to each through the galvanometer has the same direction, since on account of the opposite signs of the inductions through C_1, C_2, the currents induced in them are in opposite directions round these coils. The flow through the galvanometer due to $M\dot{u}_s$ is

$$\frac{M\dot{u}_s}{Q + \frac{PR}{P+R}} = \frac{M\frac{Q+R}{R}\dot{x}_s}{Q + \frac{PR}{P+R}} = \frac{M}{R}\frac{(P+R)(Q+R)}{Q(P+R)+PR}\dot{x}_s.$$

That due to $M\dot{x}_s$ is

$$\frac{M\dot{x}_s}{P + \frac{QR}{Q+R}} \cdot \frac{R}{Q+R} = \frac{M}{R}\frac{R^2}{P(Q+R)+QR}\dot{x}_s.$$

Adding these we get for the total flow through the galvanometer

$$x = \frac{M}{R}\frac{(P+R)(Q+R)+R^2}{(P+R)(Q+R)-R^2}\dot{x}_s$$

which agrees with (5).

Rowland's and Glazebrook's Experiments

Determinations by this method have been made by Rowland at Baltimore and Glazebrook at Cambridge. In both sets of experiments the arrangement of coils was not disturbed; but the induction-flow was produced by the simple expedient of reversing the current in the coil C_1. Rowland used a special ballistic galvanometer to measure the transient current, and a comparison of its constant with that of the galvanometer used for the steady current gave the necessary data for calculating $\dot{x}_s/x$.

Glazebrook's Experiments.

In Glazebrook's determination,* however, the same galvanometer was used for the measurement of both transient and steady currents, being shunted for the latter purpose so that only a fraction h of the current $\dot{x}_s$ produced the deflection θ_1 of the needle. Thus instead of (2) the formula of calculation

$$R = 2M\frac{\pi\tan\theta_1}{T\sin\frac{1}{2}\theta_2}\frac{1}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

was applied, the deflections of course being corrected for damping, &c. The factor 2 is introduced on the right-hand

* *Phil. Trans. R.S.* 1883.

side as the current was reversed, and therefore the induction changed by $2M$.

Details of Induction Coils.

The following are the particulars of the coils used by Glazebrook, which were wound with great care by Professor Chrystal for a similar investigation. The two coils are distinguished as A and B. They were wound with well-insulated copper wire.

	A	B	Mean.
Mean radius in cms. (a). . . .	25·753	25·766	25·760
Axial breadth of section ($2b$). .	1·896	1·899	1·897
Radial depth of section ($2d$) . .	1·92	1·90	1·91
Number of turns of wire . . .	797	791	794
Resistance (approx.) in B.A. units	84	83	167/2

The positions of the mean planes were estimated from the dimensions of the ring channels in which the wire was wound, and any doubt as to the exact positions in these channels was eliminated by reversing the bobbins relatively to the distance pieces between them.

Galvanometer.

The galvanometer used was an instrument also specially wound by Professor Chrystal. It consisted of two coils about 4 inches in diameter and 23/32 of an inch apart. These coils were movable about a vertical axis round a graduated circle could be fixed in the magnetic meridian.

The needle was of hard steel, and 1·5 cm. in length, and weighed ·708 gramme. It was suspended in a stirrup of brass on which was fixed the mirror, and a projecting stem of brass, on which brass weights were screwed to increase the period. The whole weighed 6·6 grammes, and was suspended by three fibres of silk 60 cms. long.

The scale was of paper divided to millimetres, and compared with a standard scale.

Experiments made.

Each experiment made included eight observations of throw, and two of steady current deflection, and each set of experiments consisted of four, one for each of the four positions in which the pair of coils could be placed by reversing them with-

out changing the distance between their centres. Three such sets were made for the distance 15·019 cms. of mean planes. These gave as a result

First Result.

$$1 \text{ B.A. unit} = \cdot 98598 \times 10^9 \text{ cms. per second.}$$

Three series of experiments were afterwards made in like manner for three different distances of mean planes 15·019 cms., 18·252 cms., 26·692 cms.

Different batteries were used so that the currents through the coil were varied. The mean result obtained was

$$R = 158 \times 10^9 \text{ C.G.S.}$$

As a precaution when the conductor, the resistance of which is to be determined, is a coil of copper wire, it is necessary lest the result should be affected by variation of temperature to make frequent comparisons of the resistance of the coil with that of a platinum, silver or German silver standard.

Expressed in B.A. units, R was found by such a comparison with B.A. standards to be 160·520 at 12° C., and the results reduced to this temperature for comparison gave for the B.A. unit the following values:

Final Result of Experiments.

Series A. 4 sets.

$$\cdot 98633 \times 10^9 \text{ C.G.S.}$$

Series B. 2 sets.

$$\cdot 98558 \times 10^9 \text{ C.G.S.}$$

Series C. 3 sets.

$$\cdot 98676 \times 10^9 \text{ C.G.S.}$$

Including the preliminary results, with half-weights given to them, the whole investigation gave

$$1 \text{ B.A. unit} = \cdot 986271 \times 10^9 \text{ C.G.S.}$$

Glazebrook has made a redetermination by this method of the value of the B.A. unit, and has given* as the mean of all his results

$$1 \text{ B.A. unit} = \cdot 98665 \times 10^9 \text{ C.G.S.}$$

Accuracy of Method.

In this method, apart from observations of galvanometer deflections, accuracy depends on the exact determination of M, which is a linear quantity. The coils used have had generally the same radius, and the effect of errors in the measurement of their radii and distances apart have been estimated as follows

* B. A. Report, 1890.

by Lord Rayleigh.* If we denote the mean radius of the coils (supposed the same in both) by a, and the distance apart of their mean planes by b, and $(a/M)\,dM/da$, $(b/M)\,dM/db$, by λ, μ,

$$\lambda+\mu=1$$

since M is linear, and

$$\frac{dM}{M}=\lambda\frac{da}{a}+\mu\frac{db}{b},$$

which enables the effects of the errors da/a, db/b, to be estimated.

Examination of Different Cases.

The expression for M in terms of a and b is given by (142) at p. 315 above, and the known values (see Appendix) of $M/4\pi\sqrt{aa'}$ for different values of $\gamma(=\sin^{-1}2\sqrt{aa'}/\sqrt{(a+a')^2+b^2})$, enables those of λ and μ to be found. It is clear that since M increases as b diminishes, and *vice versâ*, μ must always be negative; λ must therefore be always greater than unity.

If b be great in comparison with a it is clear that M will vary as a^4/b^3, and therefore $\lambda=4$, $\mu=-3$. This is a very unfavourable case, as then errors in a and b are unduly multiplied in M.

Again, if b be small, it is clear that μ is nearly zero, and this may be verified by differentiating the approximate expression $4\pi a\log(8a/b-2)$. Still any error db in b may, if b is small, be comparable with b itself, and thus, although μ may be small, $\mu\,db/b$ may be sensible. Further, the correction for cross-section is of greater relative importance in this case; and thus for two reasons it is preferable to keep b of moderate value. Lord Rayleigh gives the following table for intermediate values of b:—

γ	$b/2a$	λ	μ	M
60°	·577	2·61	− 1·61	·316
70°	·364	2·18	1·18	·597
75°	·268	1·98	− ·98	·829
80°	·176	1·76	− ·76	1·186

This table shows that for equal values of da/a, and db/b, the numerical values of the errors in M are roughly as 2 to 1.

* *Phil. Mag.* Nov. 1882.

Method nearly Independent of Measurement of Scale Distances.

With regard to the current measurements, it is to be noticed that the method does not involve any determinations of distances of scales from mirrors, except as a means of correcting the approximate value of $\tan\theta_1/\sin\frac{1}{2}\theta_2$ given by the ratio of the deflections as read off in scale divisions (see p. 486).

Lord Rayleigh is of opinion that by using still larger coils than those employed by Glazebrook, with the same number of turns of wire, the accuracy of experiments by this method might probably be still further increased. The greater value of M, and the greater conductance of the wire, would give greater sensibility, and the linear measurements could be more exactly made. A relatively small value of the radial breadth of section, the chief element in the correction for cross-section, might then also be used.

Rowland's Experiments. Details of Induction Coils.

The induction coils used in Rowland's experiments* were made by winding 154 turns of fine silk-covered wire in each of three accurately turned brass bobbins (A, B, C). Their mean radii were respectively 13·710 cms., 13·690 cms., 13·720 cms., and each had a radial depth of ·90 cm. and an axial width of ·84 cm.

These bobbins were used two at a time, and were made with carefully ground ends so that they could be fitted end to end with their axes in line. Each pair could of course be placed in four positions relatively to one another without altering the distance between their mean planes, and as all four were used in each case, the slightest uncertainty as to the exact distance of the coils apart was eliminated by combination of the results. The distance of the bobbins was measured for each position by means of a cathetometer applied at three different points in the circumference.

Value of M.

The values of M were calculated by the elliptic integral formula already given, and a correction was made for the cross-section of each coil according to the formula at p. 522 above. The results were as follows:—

	A and B.	A and C.	B and C.
Mean distance apart	6·534 cms.	9·574 cms.	11·471 cms.
Value of M . .	3775500 cms.	2561974 cms.	2051320 cms.

* Silliman's American Journal, 15 (1878).

The ballistic galvanometer was composed of two coils containing between them 1,790 turns of No. 22 silk-covered copper wire, wound on a brass cylinder 8·2 cms. long, and 11·6 cms. in diameter, in rectangular grooves 3 cms. deep and 2·5 cms. wide. A saw-cut along the cylinder prevented the circulation of induction currents round it. The coil was mounted so that it could be turned about a vertical axis to any required azimuth, and its position determined by a horizontal circle below. This circle was finely graduated, and was read to 30″ by a couple of verniers. Details of Ballistic Galvanometer.

Two different needles were used in each, consisting of two thin laminæ of hard steel attached to the two sides of a square piece of wood so that the magnetic axis could not vary in position. One needle was 1·27 cms. long, and had a period of 7·8 seconds; the length of the other was 1·20 cms., and its period 11·5 seconds. The moment of inertia of each was augmented by brass weights carried by wires extending in the direction of the magnetic axis. Each needle was suspended by three single fibres 43 cms. long. The torsion of these fibres was eliminated from the result, as will be seen below, except as regarded the period of vibration, and for this an allowance was made. Needles of Galvanometer.

A brass bar, passing through the opening below the needle, carried a small telescope by which the mirror was observed when the constant of the coil was compared with that of another.

The constant, G_2, of the coil was determined first by calculation from its dimensions, and by comparison with that of the large double coil of an electrodynamometer constructed on Helmholtz's plan (p. 365 above). This coil had a constant of 78·37 by calculation. In the comparison the ballistic galvanometer was used with its graduated horizontal circle as a sine galvanometer. Determination of Galvanometer Constant.

After a comparison had been made the instruments were interchanged, and the comparison repeated to eliminate the ratio of the values of H at the two places.

Seven determinations gave as a mean result $G_2 = 1833·67$, with a probable error of $\pm·09$, and calculation gave $G_2 = 1832·24$. The former result, being probably considerably the more accurate, was given double weight, and a mean then taken with the latter, which gave $G_2 = 1833·19$.

A tangent galvanometer was used to measure the steady current. This was a circle 50 cms. in diameter, and had a needle 2·7 cms. long, the deflection of which was read by a pointer moving round a graduated circle 20 cms. in diameter. Parallactic error was avoided by placing the circle on a level with the needle which moved round inside it. Details of Tangent Galvanometer.

N N 2

Elimination of Ratio H/H'.

The constant of this galvanometer was compared with that of a single circle of wire 82·7 cms. in diameter, wound on a ring made of pieces of wood laid together with the grain in the direction of the circumference, and carefully turned with a small groove near one side to receive the wire. The length of the wire was 259·58 cms., giving a mean radius of 41·31344 cms. This circle was made to surround the ballistic galvanometer coil, but at a distance of 1·1 cm. on one side, to allow the tube carrying the suspension fibre to pass. Thus the constant of the circle was ·151925.

The same current being sent through the tangent galvanometer coil and the ring, and G_1, G', being their respective constants, we have, if a, a', be the angular deflections of the needles,

$$\frac{H}{G_1}\tan a = \frac{H'}{G'}\tan a',$$

so that

$$\frac{H}{H'} = \frac{G_1}{G'}\frac{\tan a}{\tan a'},$$

Equation for R.

and this replaces H/H' in (2′), which becomes

$$R = M\frac{\pi}{T}\frac{G_2}{G'}\frac{\tan a'}{\tan a}\frac{\tan\theta_1}{\sin\frac{1}{2}\theta_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where θ_2 is the ballistic deflection corrected for damping.

Advantages of Procedure adopted.

This method avoids the difficulty of accurately determining H/H' by vibration of a needle at the two places, and gives the further great advantage that the distance of the mirror from the scale of the ballistic galvanometer only enters as a correction on the ratio $\tan a'/\sin\frac{1}{2}\theta_2$. The same factor of correction for torsion affected both $\tan a'$ and $\sin\frac{1}{2}\theta_2$, so that, with the exception of a small correction on the period T of the needle of the ballistic galvanometer, all allowances for torsion were eliminated. Still further, since a and θ can be made nearly equal, the correction for length of needle in $\tan\theta/\tan a$ is almost entirely obviated.

The apparatus was set up in a separate building in two rooms on the ground floor. The galvanometers were on brick piers, with marble tops, and were very carefully adjusted, and all connecting wires were twisted together to avoid magnetic effect. This adjustment, as well as the insulation everywhere, was carefully tested.

Experiments made.

The experiments were mainly made by simply reversing the battery current and observing the throw, but the method of recoil was also used. Series of experiments were made with each pair of induction coils A and B, B and C, C and A.

The time of vibration was observed at the beginning and end of each series of observations. The needle was allowed to vibrate for 10 seconds, and ten observations were made before and after that interval. Time was taken on an accurate marine chronometer.

Absolute value of R.

The mean result of a long series of experiments gave, after all corrections for temperature of coils, &c., $34{\cdot}719 \times 10^9$ cms. per sec. as the value of R. Comparing with "10 ohm" standard coils in his possession, and with a resistance box by Elliott, Professor Rowland came to the conclusion that

Rowland's Final Result.

$$1 \text{ B.A. unit} = {\cdot}9911 \times 10^9 \text{ cms. per sec.}$$

Weber's Earth Inductor Method.

Two methods of the first class are due to W. Weber. The first is very simple. A coil mounted with its axis of figure horizontal and in the magnetic meridian, and having its circuit completed through a ballistic galvanometer, is quickly turned through half a revolution round a vertical axis. If A be the effective area of the coil (the sum of the areas of its spires), and H the horizontal component of the earth's field-intensity, a change of induction of amount $2AH$ through the coil is produced. This measures the integral electromotive force in the coil, and hence if the circuit be completed, and include a total resistance R, the total quantity of electricity which flows through the circuit is $2AH/R$. This is not affected in the least by the inductance of the circuit.

Theory of Method.

The galvanometer deflection is observed, and also the elongations following, to allow damping to be corrected for. By the theory of the ballistic galvanometer, if T be the complete period of the needle, G the principal

galvanometer constant, H' the horizontal component of the earth's magnetic field *at the needle*, and θ the observed deflection, the total flow through the instrument is $HT \sin \frac{1}{2}\theta / \pi G$. Thus

$$\frac{2AH'}{R} = \frac{HT}{\pi G} \sin \tfrac{1}{2}\theta$$

or

$$R = 2\pi GA \frac{H'}{H} \frac{1}{T \sin \frac{1}{2}\theta} \quad . \quad . \quad . \quad (8)$$

In general H is very nearly equal to H', but it will not do to assume absolute equality; and the two quantities must be compared by observing the periods of vibration of a horizontally suspended needle at the two places.

Weber's Mode of Experimenting

Weber employed the method of recoil (p. 396 above) in his observations. Turning the coil first through 180° from the initial position, he observed one deflection (positive, say) and the following elongation. Then when the needle was passing through zero the second time, he brought the coil back to its original position. This brought the needle to rest, and finally deflected it to the negative side of zero. This deflection was observed, and the following elongation, and then, at the second passage through zero, the same series of operations was begun afresh.

Criticism of Method.

Lord Rayleigh has pointed out that if a, a', be the mean radius of the inductor and galvanometer coils respectively, the product

$$GA = 2\pi^2 \frac{a^2}{a'},$$

so that error of mean radius has double the importance in the inductor coil that it has in the galvanometer.

Great care is necessary in levelling the inductor as, on account of the largeness of the vertical component of the earth's field in high latitudes, any deviation in the plane of the meridian of the axis of rotation from verticality will lead to error of the same order in the result. Thus if the axis be inclined to the vertical

at a small angle a in the plane of the meridian, we must use instead of A the value $A(1 + a \tan D)$, where D is the magnetic dip.

Weber and Zollner's Experiments.

This method was used by Weber himself, and later by Weber and F. Zollner. In the latter experiments very large inductor and galvanometer coils were used. Each consisted of 12 layers of copper wire 3 mm. thick, 66 turns in a layer, wound on bobbins of well seasoned, oil-soaked mahogany. The dimensions were:—

	Int. Radius.	Ext. Radius.	Length.
Inductor . . .	48·0414 cms.	51·9461 cms.	25·420 cms.
Galvanometer .	48·032 cms.	52·0797 cms.	25·420 cms.

For the galvanometer needle was used one or other of two magnets of lengths 10 cms and 20 cms. respectively, and the deflections were read by means of a telescope and scale in the ordinary manner. The research was carried out in a room of the observatory at Leipzig, subject to varying magnetic disturbances and to variations of temperature, and was intended merely as a test of the apparatus.

The resistance of the circuit of the inductor given by the experiments came out slightly greater with the shorter needle than with the other. This was to be expected as the deflection, θ, with the shorter magnet must, on account of the greater distance on the whole of its magnetic distribution from the current, have been slightly smaller than the deflection in the other case. It is obvious that the needles were much too long.

Wiedemann's Experiments.

A careful determination of the ohm has been made with these coils by Professor G. Wiedemann.* The apparatus was set up in a room of very constant temperature in the University of Leipzig. A rhombus-shaped steel plate, with attached glass mirror, was hung with its plane vertical and its longest diameter horizontal, and being magnetized in the direction of this diagonal served as needle. The needle carried beneath it a horizontal metal bar on which weights could be slided to alter the moment of inertia of the suspended system.

Adjustment of Inductor and Galvanometer Coils.

The coils, having been levelled, were each adjusted until the same current sent in opposite directions produced equal deflections of a needle hung within the coil. Their axes were then at right angles to the magnetic meridian. The galvanometer coil

* *Abhandl. Berlin Akad. der Wissensch.* 1884, or Wiedemann's *Elektricität*, Band 4, p. 913.

was then fixed, and the inductor turned through an angle of 90°. This angle was measured by means of a right angled glass prism, by observing a telescope scale by reflection in one of the rectangular faces (which were vertical), and turning the coil until the same division came to the cross-wires by reflection from the other face.

Method of Multiplication.

An arrangement of stops was then provided so that the coil could be turned from this position through exactly 180° and back again. The coil was turned a number of times in succession suddenly through this angle, always when the needle had returned to its zero position, so that the deflection was multiplied as far as the limits of the scale would allow.

The successive deflections θ_1, θ_2, &c., if the current was applied when the needle was accurately at zero in each case, were related to the quantity Q of electricity which flowed through the circuit at each half turn of the coil as follows: $\theta_1 = KQ$, $\theta_2 = KQ\epsilon^{-\lambda}$, $KQ(1-\epsilon^{-2\lambda})\,Q$, . . ., where K has the value stated in (39), p. 396 above. These were observed and the observations combined in a single formula for Q, which equated to $2AH/R$ enabled R to be calculated.

Determination of Ratio H'/H.

The periods T, T', of a needle vibrated at the galvanometer and inductor respectively were observed, and the ratio T^2/T'^2 gave the value of H'/H required as shown in 8,. These were obtained by observing the oscillations with a telescope and scale, and registering the passages of different points of the scale across the wires by means of a chronograph.

The effect of torsion of the suspension fibre was found by turning a torsion head, to which the fibre was attached, through a measured angle, and observing the corresponding deflection of the needle. Thus when the torsion head was turned through an angle α, and the needle through an angle β, the return couple on the needle was $MH\sin\beta$, and the torsional couple $C(\alpha - \beta)$, where C is a constant. Thus

$$C = \frac{MH\sin\beta}{\alpha - \beta} = MH\tau, \text{ say.}$$

Hence, when the needle in the experiments was deflected through an angle θ, the return couple upon it was $MH(\sin\theta + \tau\theta)$, or nearly enough, as the deflections were small, $MH(1 + \tau)\theta$. Thus instead of the value of H at the galvanometer needle was used $H(1 + \tau)$.

Measurement of Coils.

The dimensions of the coils were measured by determining their inner and outer circumferences with a steel tape, and as a check by measuring three diameters at intervals of 60° apart, by means of a cathetometer.

The distance of the scale from the mirror was first measured by means of a steel tape on which were sliding pieces furnished with points, which were brought against the mirror and scale respectively; then, by means of an auxiliary scale placed horizontally in the vertical plane through the centres of the telescope and mirror, on which the corresponding positions of the mirror and reading scale were observed by means of a cathetometer. Measurement of Distance of Scale.

Experiments were made first with Weber and Zöllner's coils in the state in which they were left by these experimenters; then with the same coils rewound, and the number of turns increased from 792 to 804.

The experiments were then repeated with 10 mercury (Siemens) units included with the coils in the circuit.

Different series were made with the sliding weights on the needle at distances 2 cms., 1·5 cm., 1 cm., 0, from the end of the bar, so that the periods were altered through a considerable range.

The resistance of the Siemens' units was compared with a standard resistance of pure mercury, consisting of a mercury column contained in a carefully calibrated tube 106·398 cms. long, the ends of which communicated with electrodes made of amalgamated copper-foil immersed in mercury in two vessels terminating the tube. It was found as a final mean result that 1 ohm or 10^9 C.G.S. units of resistance is equal to the resistance at 0° of a column of mercury 106·162 cms. long and 1 sq. mm. in cross-section. Results of Wiedemann's Experiments.

This method has also been used by Mascart, De Norville, and Benoit, in a very elaborate series of experiments. Five coils were used, two of 27 cms. internal and 30 cms. external diameter, and 3 cms. length, and three smaller coils each of 14 cms. internal and 17 cms. external diameter, and the same length as before. These were wound with silk-covered wire 5 mm. in diameter. One of the large coils and two of the small ones were wound with separate layers, so that, by joining these layers up differently, nine different arrangements could be obtained. The winding was performed with the wire under tension produced by passing it over loaded rollers when on its way from the reel to the bobbin. The length of the wire was measured as it was laid on, and the diameter of every turn was also observed by means of callipers. Experiments of Mascart, de Nerville, and Benoit.

Both the smaller and larger coils were mounted after completion on stands with suitable stops so as to admit of being turned when required through an angle of exactly 180°, and were set up with their axes horizontal and in the magnetic meridian. Arrangement of Apparatus.

At the centre of the larger coil when in position was placed a

small magnetometer needle suspended by a single fibre of silk. By turning the coil round a vertical axis through 90° from its position when arranged for inductive use, and fixing it in its new position, it could be used as a galvanometer bobbin, and its galvanometer constant compared with that of the galvanometer bobbin itself. By this process, previously used by Rowland, the ratio of the horizontal magnetic forces, H'/H at the inductor and the galvanometer was eliminated from the formula of calculation. For suppose the same current to be sent through the two coils, and α, α', to be the deflections for the galvanometer and the inductor respectively, G, G', the galvanometer constants of the two coils, we have, as at p. 548,

Elimination of Ratio H'/H.

$$\frac{H'}{H} = \frac{G'}{G}\frac{\tan\alpha}{\tan\alpha'}.$$

This substituted in (8) gives

$$R = 2\pi G'A\frac{\tan\alpha}{\tan\alpha'}\frac{1}{T\sin\frac{1}{2}\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

[Full details of the mode of comparing two galvanometer constants are given at pp. 405, 406 above.]

Approximate Elimination of Scale Distance. This proceeding had the advantage (already pointed out p. 546) that since the ratio of $\tan\alpha/\sin\frac{1}{2}\theta$ appears in the value of R the importance of an exact determination of the distance of the galvanometer scale from the mirror was greatly lessened. The value however of $\tan\alpha'$ had to be accurately known, and involved careful measurement of the corresponding distance for the other scale.

Evaluation of Product of Constants of Coil. From the measured length of the wire the value of $G'A$ which appears in (9) could be approximated to. For a being the mean radius of the coil, and n the number of turns $A = n\pi a^2$, and $G' = 2n\pi/a$, nearly, so that $G'A = 2n^2\pi^2a = n\pi l$ where l is the length of the wire. The quantities therefore which required accurate determination were l, T, and the distance of the scale from the mirror of the magnetometer in the induction coil. The latter was found by means of a graduated measuring bar carrying sliding pieces, which were run up to the fibre and scale respectively. The positions of the contact faces of these pieces were read off from the scale and gave the distance required.

Observations. Observations were made by first reading off two successive elongations of the needle when it had nearly come to rest, and then turning the inductor when the needle was passing through zero, and reading the following elongations on the same side of zero.

Reduction of Observations.

If r, r', be the two readings on the scale (supposed graduated from one end), the zero reading is $(r'+r)/2$. If the next two readings be r_1, r_2 the first deflection from zero is $r_1 - (r'+r)/2$. The next reading being r_3 the diminution in one swing due to damping is $(r_1 - r_3)/2$. The diminution of the first elongation must have been approximately $\frac{1}{2}$ of this or $(r_1 - r_3)/4$. This correction applied to the first elongation gives for the deflection $r_1 - (r'+r)/2 + (r_1 - r_3)/4$. There remains the correction for the initial motion which (r' being taken as the greater reading) would have carried the needle through a deflection of $\pm(r'-r)/2$, according as the initial motion was with or against the induction throw. Thus the deflection was

$$r_1 - \frac{r'+r}{2} + \frac{r_1 - r_3}{4} \pm \frac{r'-r}{2}$$

The readings it was found did not vary more than $\frac{1}{8}$ per cent.

Elimination of Torsion of Fibre.

The torsion of the suspension fibre of the ballistic galvanometer was eliminated, as approximately it multiplied $\tan a$ and $\sin \frac{1}{2}\theta$ in (9) by a common factor. That of the suspension fibre of the inductor was determined as described above (p. 364) by turning the upper end of the fibre round through 360°.

Experiments were made with the various coils arranged in different ways; and their effective areas were also compared by observing the effects which they produced on the galvanometer needle when turned in the earth's field.

Comparison of Absolute Resistance with B.A. Units.

The absolute resistance of the circuit in the various experiments having been obtained it was compared by Carey Foster's method with four B.A. units, with four Siemens' mercury units, and with six specially constructed mercury units in spiral tubes. Careful comparisons of the temperature coefficients of the different coils were made, and all the resistances corrected to the temperature of experiment. The results were expressed finally as the absolute resistance of four mercury standards made of carefully calibrated tubes filled with mercury. These tubes were terminated by wide electrodes of mercury, and an allowance of a length of the tube equal to ·82 of its diameter was made to correct for the additional resistance due to the abrupt change of section of the tube at each end. The final result obtained was

$$1 \text{ ohm} = 1{\cdot}0142 \text{ B.A. unit,}$$

or

1 ohm = resistance at 0° C. of a column of mercury 106·37 cms. long and 1 sq. mm. in section.

Weber's Method by Damping. Weber's second method consists in oscillating a magnet suspended within a coil, when the circuit is open, and again when the circuit is closed, and observing the period and logarithmic decrement in both cases. The induced currents assist the damping in the second case, and hence from a comparison of the results the resistance of the coil can be calculated.

Theory of Method. When the circuit is open the equation of motion of the swinging needle is

$$\frac{d^2\phi}{dt^2} + 2k\frac{d\phi}{dt} + \frac{MH}{\mu}\phi = 0 \quad . \quad . \quad . \quad . \quad (10)$$

where M is the magnetic moment, H the horizontal field intensity, and μ the moment of inertia of the magnet. Putting n^2 for MH/μ we get for the solution of the equation

$$\phi = Ae^{-kt}\cos(\sqrt{n^2 - k^2}\,t + \epsilon) \quad . \quad . \quad . \quad . \quad (11)$$

Here $k = 2\lambda/T$ if λ be the logarithmic decrement and T the observed period $(= 2\pi/\sqrt{n^2 - k^2})$.

Theory of Method by Damping. If now the circuit be parallel to the meridian and be closed, the magnet will be acted on by the induced current produced by its motion. The magnetic induction through the coil due to the needle is $MG \sin\phi$ approximately, where G is the principal galvanometer constant of the coil. For let a current γ flow in the coil, then the mutual energy of the coil and magnet is equal to the product of the magnetic induction of the magnet through the coil and the current. But when $\phi = 0$ this energy is obviously zero and the work done against the current in deflecting the magnet through the angle ϕ is $MG\gamma \sin\phi$, and the magnetic induction through the circuit is $MG \sin\phi$. Supposing then the magnet swinging through a small range there will be a force exerted on the magnet by the current of amount $MG\gamma$. Hence the equation of motion of the magnet is

$$\frac{d^2\phi}{dt^2} + 2k\frac{d\phi}{dt} + n^2\phi - \frac{MG}{\mu}\gamma = 0 \quad . \quad . \quad . \quad (12)$$

But we have also for the electromotive force in the circuit $-MGd\phi/dt$ and if L be the self-inductance of the coil

$$L\frac{d\gamma}{dt} + R\gamma + MG\frac{d\phi}{dt} = 0 \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Operating on equation (12) by $L d/dt + R$, and on (13) by MG/μ, and adding, we eliminate γ, and find

$$\left(L\frac{d}{dt}+R\right)\left\{\frac{d^2}{dt^2}+2k\frac{d}{dt}+n^2\right\}\phi+\frac{M^2G^2}{\mu}\frac{d\phi}{dt}=0. \quad (14)$$

If we suppose that the motion is simple harmonic with diminishing range, and put λ', T', for the logarithmic decrement and period we may write conveniently for our present purpose

$$\phi = \epsilon^{-(k'+ia)t}$$

where $i = \sqrt{-1}$, $k' = 2\lambda'/T'$, $a = 2\pi/T'$. Thus we find

$$\frac{d}{dt} = -(k'+ia)$$

and (14) becomes

$$\{-(k'+ia)L+R\}\{k'^2-a^2+2ik'a-2k(k'+ia)+n^2\}$$
$$-\frac{M^2G^2}{\mu}(k'+ia)=0. \quad (15)$$

The real and imaginary parts of this equation must vanish separately, and therefore picking out the imaginary terms equating them to zero, and solving for R we obtain

$$R=\frac{M^2G^2}{2\mu(k'-k)}+\tfrac{1}{2}L\left(3k'-k+\frac{n^2-k^2-a^2}{k'-k}\right). \quad (16)$$

A controlling equation is obtained in like manner from the real terms in (15).

This method has been used by W. Weber himself, and with modifications by H. F. Weber, Dorn, Wild, and F. Kohlrausch. It is against the method that M^2, G^2, enter to the second power, inasmuch as the very exact determination of either quantity is a matter of some difficulty. The value of μ also involves the square of the dimensions of the magnet.

Kohlrausch's Modification of Method by Damping.

The modification of the method used by Kohlrausch amounted to a combination of the first and second methods of Weber, in which he eliminated the constant of the galvanometer with which the earth-inductor was connected by determining the logarithmic decrement of the motion of the needle first when the circuit of the galvanometer was open, and again when it was closed. Calling these decrements λ_0, λ, and putting α, β,

for the arcs of vibration in the method of recoil (which was used) T_0 the period of the needle when the circuit was open, we may write Kohlrausch's formula in the approximate form

$$R = \frac{16 A^4 H^2 T_0 (\lambda - \lambda_0)}{\pi^2 \mu} \frac{\alpha\beta}{(\alpha^2 + \beta^2)^2}.$$

Kohlrausch's Result. This formula includes several quantities which are difficult to observe with accuracy, but its chief defect lies in the fact that it involves the fourth power of the radius of the inductor. Kohlrausch's final result, corrected for an error in the data used in his original calculations, is

$$1 \text{ B.A. unit} = 990 \times 10^6 \text{ C.G.S.}$$

Method of Revolving Coil. Another method of this class suggested by Sir William Thomson to the Committee of the British Association seems also to have been first proposed by Weber. It consists in spinning with uniform velocity about a vertical axis, a circular coil at the centre of which is suspended a small magnetic needle. A periodic current is thus made to flow in the coil in one direction (relatively to the coil) in one half-turn from a position at right angles to the magnetic meridian, and in the opposite direction in the next half-turn. But the position of the coil being reversed in every half-turn as well as the current in it, the current flows on the whole in the same average direction relatively to the needle and (apart from self-induction) has its maximum value always when the plane of the coil is in the magnetic meridian.

This method was used by the British Association Committee in their famous experiments, carried out principally by Clerk Maxwell, Balfour Stewart, and Fleeming Jenkin in 1863. Its theory was first fully

given by Maxwell, and the following statement follows on the whole his notation and method.

Theory of Method of Revolving Coil.

If L be the self inductance, γ the current at any time t, the electro-kinetic energy of the circuit due to its own induction is $\frac{1}{2}L\gamma^2$. Again if M be the magnetic moment of the needle, and G the galvanometer constant of the coil, that is, the magnetic force at the centre which unit current in the coil would produce, the magnetic force at the needle due to the current γ is $G\gamma$. If ϕ be the angle which the axis of the needle makes with the magnetic meridian, and θ the angle which the coil makes with the same plane, the direction of the magnetic force due to the coil and the axis of the needle are inclined at an angle $\pi/2 - (\theta - \phi)$. Thus the mutual energy of the needle and current is numerically $MG\gamma \sin(\theta - \phi)$. This if taken as potential energy must be written with the positive sign, and if taken as kinetic energy with the negative sign prefixed to give the corresponding force. For the magnet is deflected in the direction of rotation, and hence, if $\theta > \phi$ say, the magnetic force on the needle due to the coil must be in the direction to increase ϕ, that is to diminish $\theta - \phi$. Hence $MG\gamma \sin(\theta - \phi)$ tends to diminution by the action of the mutual forces. We shall reckon it as kinetic energy of amount $- MG\gamma \sin(\theta - \phi)$.

Again if the effective area of the coil be A, there is mutual energy between it and the field of numerical amount $AH\gamma \sin\theta$. This may be taken as kinetic energy of amount $- AH\gamma \sin\theta$. Also the magnet is deflected in the field, and therefore between it and the field there is mutual energy $MH\cos\phi$ when reckoned as kinetic.

Lastly if mk^2 be the moment of inertia of the needle about the axis of suspension it has kinetic energy $\frac{1}{2}mk^2\dot{\phi}^2$.

Collecting these terms we get for the total kinetic energy

Total Kinetic Energy of Currents Needle, &c.

$$T = \tfrac{1}{2}L\gamma^2 - AH\gamma \sin\theta - MG\gamma \sin(\theta - \phi) + MH\cos\phi + \tfrac{1}{2}mk^2\dot{\phi}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Potential Energy of Torsion of Fibre.

Besides this there is potential energy V, due to the torsion of the fibre, depending on the angle through which the needle has been turned from the position of no torsion. If α be the angle which the needle makes with the meridian when the torsion is zero, the angle through which the fibre has been turned is $\phi - \alpha$. Denoting by $MH\tau$ the torsional couple which

the wire gives when the lower end is turned through unit angle relatively to the upper, we have

$$V = \int_a^\phi MH\tau(\phi - a)d\phi = \tfrac{1}{2}MH\tau(\phi - a)^2 \quad . \quad . \quad (18)$$

Differential Equation of Currents. The equation of currents is

$$\frac{d}{dt}\frac{\partial T}{\partial \gamma} + \frac{\partial F}{\partial \gamma} = 0$$

where F is the dissipation function. This gives by (17)

$$L\frac{d\gamma}{dt} + R\gamma = AH\dot{\theta}\cos\theta + MG(\dot{\theta} - \phi)\cos(\theta - \phi). \quad (19)$$

There are two possible distinct motions for the magnet, one of oscillation in its own proper period (which we suppose great in comparison with the period of rotation of the coil), and the other of period equal to half that of rotation. So far as the former is concerned, we may take the magnet as at rest in computing the current, and for the latter we shall suppose at present the amplitude very small, so that the part of ϕ depending upon it may also be neglected and ϕ may be taken as constant. Thus $\dot{\theta}$ being constant $= \omega$, say, and $\theta = \omega t$, we have

$$L\frac{d\gamma}{dt} + R\gamma = AH\omega\cos\omega t + MG\omega\cos(\omega t - \phi) \quad . \quad (20)$$

Let a solution of this equation be

$$\gamma = C\cos\omega t + C'\sin\omega t.$$

Then

$$L\frac{d\gamma}{dt} + R\gamma = (L\omega C' + RC)\cos\omega t - (L\omega C - RC')\sin\omega t \quad (21)$$

Integral Equation of Current. This with (19) gives by equation of coefficients

$$\gamma = \frac{\omega}{R^2 + \omega^2L^2}\left[AH(R\cos\theta + \omega L\sin\theta) + MG\{R\cos(\theta - \phi) + L\sin(\theta - \phi)\}\right] \quad . \quad (22)$$

A term, $C \exp(-Rt/L)$, is required to complete the solution, but this dies out soon after the starting of the coil, and has no effect provided the rotation is uniform. The current therefore on the supposition made is given by (22).

The expression for the kinetic and potential energies gives for the equation of motion of the magnet

Equation of Motion of Magnet.

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} + \frac{\partial V}{\partial \phi} = 0$$

or $$mk^2\ddot{\phi} - MG\gamma\cos(\theta-\phi) + MH\sin\phi + MH\tau(\phi-\alpha) = 0 \qquad (23)$$

This equation may be obtained also by considering that the needle is acted on by three couples, one due to the current tending to produce further deflection, the second a return couple due to the earth's magnetic field, and the third also a return couple due to the torsion of the fibre. The numerical values of these are from the notation already explained, $MG\gamma\cos(\theta-\phi)$, $MH\sin\phi$, $MH\tau(\phi-\alpha)$. Hence the total deflecting couple is

$$MG\gamma\cos(\theta-\phi) - MH\{\sin\theta + \tau(\phi-\alpha)\}$$

and this is equal to the rate of increase $mk^2\ddot{\phi}$ of angular momentum.

The needle is found to take up a nearly constant position if the rotation of the coil is kept uniform, and in this case $\dot{\phi}$ may be taken as very nearly zero. Thus we have, integrating over any finite interval of time, $\int \ddot{\phi}\,dt = 0$. The mean resultant deflecting couple applied by the current must therefore be equal to the return couple $MH\{\sin\phi + \tau(\phi-\alpha)\}$ due to the combined action of the magnetic field and torsion. This average couple is obtained from $MG\gamma\cos(\theta-\phi)$ by inserting the value of γ given by (22) and integrating each term over a whole turn on the supposition that ϕ is a constant, and dividing the result by 2π. The following integrals enter into the expression

$$\frac{1}{2\pi}\int_0^{2\pi}\cos\theta\cos(\theta-\phi)d\theta = \tfrac{1}{2}\cos\phi.$$

$$\frac{1}{2\pi}\int_0^{2\pi}\sin\theta\cos(\theta-\phi)d\theta = \tfrac{1}{2}\sin\phi.$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2(\theta-\phi)d\theta = \tfrac{1}{2}.$$

$$\frac{1}{2\pi}\int\sin(\theta-\phi)\cos(\theta-\phi)d\theta = 0$$

Quadratic Equation for Resistance of Coil-Current.

Therefore the average couple is

$$\frac{1}{2}\frac{\omega MG}{R^2 + \omega^2 L^2}\{AH(R\cos\phi + \omega L\sin\phi) + MGR\},$$
$$= MH(\sin\phi + \tau\phi) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

for equilibrium, if $a = 0$. Since τ is very small we may write $\tau \sin\phi$ for $\tau\phi$, and we get

$$R^2 - \frac{1}{2}R\frac{\omega AG}{1+\tau}\cot\phi\left(1 + \frac{MG}{AH}\sec\phi\right) + L^2\omega^2$$
$$- \frac{1}{2}\frac{AGL\omega^2}{1+\tau} = 0 \quad . \quad . \quad . \quad . \qquad (24')$$

This may be written in the form

$$R^2 - aR - b = 0$$

the solution of which is

$$R = \frac{a \pm \sqrt{a^2 + 4b}}{2}$$

Resistance developed in Series.

The value of b is positive in the experiments made, and hence, since R cannot be negative, the $+$ sign in the solution must be taken. Expanding the radical, having regard to the fact that MG/AH and τ are small, we get

$$R = \frac{1}{2}AG\omega\cot\phi\left\{1 - \frac{2L}{AG}\left(\frac{2L}{AG} - 1\right)\tan^2\phi\right.$$
$$\left. + \left(\frac{2L}{AG}\right)^2\left(\frac{2L}{AG} - 1\right)^2\tan^4\phi - \ldots\right\} \quad . \quad . \quad . \qquad (25)$$

This is the expression for R used by the B.A. Committee in the reduction of the results of their experiments.

Criticisms of Method.

Taking the first term only we may write

$$R = \frac{1}{2}AG\omega\cot\phi = \pi^2 n^2 a\omega\cot\phi \quad . \quad . \quad . \qquad (26)$$

where a denotes the mean radius and n the number of turns in the coil. This formula is convenient for the discussion of the advantages and disadvantages of the method. These have been

examined by Lord Rayleigh in papers on this method * and in his "Comparison of Methods for the Determination of Resistances in Absolute Measure." †

As regards the measurement of dimensions of the apparatus, it is to be noticed that the method involves only one fundamental linear quantity a, and that only to the first power. The observation of the deflection corresponding to ϕ and the evaluation of $\cot\phi$ involve no greater difficulty than those involved in ordinary angular measurement, and in this respect the method is on a par with Weber's method by earth inductor. The main difficulties lie in the determination of ω and the avoidance of mechanical disturbance, and of error due to currents in the ring and alterations in the magnetization of the needle.

Stroboscopic Method of Measuring Speed.

It will be seen below that, by the employment of what may be called the stroboscopic method of observation, Lord Rayleigh, who repeated the determination with the same apparatus, was able to control and measure the speed with great exactness. A correction is easily made for the currents induced in the coil in consequence of its motion in the field of the needle, in fact a small term appears in the result above [$MG \sec\phi/AH$ in (24)], by means of which this correction is made. This involves the determination of MG/AH, but, as will be seen below, about this there is no difficulty whatever.

Correction for Ring Currents.

The currents produced in the metal ring can be allowed for by rotating the coil (1) with the wire circuit open, (2) with that circuit closed. Further, these currents can be reduced by dividing the ring into two parts along a diameter and putting them together with ebonite separating pieces. The currents are then confined to circuits which are on the whole at right angles to the plane of the coil, and their effect can easily be eliminated by the method just stated. The existence of these currents in the ring has one advantage pointed out by Lord Rayleigh, that by rotating the coil before winding, and again with the wire circuit open after winding, the insulation can be tested. For if any difference is found between the deflections of the needle it must be due to leakage.

The method has been objected to on the ground of the influence of self-induction in the result, that is on account of the terms in (25) which involve L. Now the value of the

* Lord Rayleigh and Arthur Schuster, "On the Determination of the Ohm," *Proc. R.S.* No. 213, 1881. Lord Rayleigh, *Phil. Trans. R.S.* Part ii. 1882

† *Phil. Mag.* Nov. 1882.

coefficient (U, say) of $\tan^2\phi$ in (25), and therefore of $\tan^4\phi$, &c., may be calculated with considerable accuracy from the dimensions and arrangement of the rotating coil, and any want of exact knowledge of the value of U can be eliminated by using different speeds of rotation.

Comparison of Earth-Inductor and Revolving Coil Methods. Effect of Self Induction.

In comparing Weber's method by earth inductor with the present method, it is to be noticed that at half the lowest speed used by Lord Rayleigh the sensitiveness of the former method would be considerably less than that of the latter, and the correction for self-induction, known with fair accuracy, would be only about $\frac{1}{4}$ per cent.

The effect of self-induction could be diminished, as pointed out by Lord Rayleigh, by duplicating the revolving coil by the addition of a second coil at right angles to the other, and giving an independent circuit. Thus the sensitiveness of the arrangement would be increased without entailing an increased correction for self-induction such as would be necessary if the increase of deflection were produced by running the coil at a higher speed. The two circuits in this arrangement also would be conjugate, that is the currents in one would be unaffected by those in the other, and would give a more nearly constant field of magnetic force.

Dimensions of Revolving Coil.

We now give some account of later determinations by this method, beginning with the experiments made by Lord Rayleigh and Prof. Schuster in 1881.* The coil used by the B.A. Committee was employed, but its constants were carefully redetermined. The constant A of the coil was found by unwinding the wire, and carefully measuring the circumference of the successive layers. The thickness of the wire used was 1·37 mm., which ought to have produced a difference in the circumference of the successive layers of $2{\cdot}74\pi$ mm. The turns in each layer sinking a little into those below gave on the average 8·1 mm. for this difference. On each coil there were 156·5 turns arranged in one case in 12 layers of 13 turns each, with half a turn outside, and in the other in 12 layers containing 155 turns with $1\frac{1}{2}$ turns outside. Allowing for the outside parts these measurements gave

Mean radius of double coil . . .	15·789 cms.
Axial dimension of each groove . .	1·833 cm.
Distance of mean plane from axis of motion	1·918 cm.

* *Proc. R.S.* No. 213, 1881.

The value of A was calculated by the formula

Dimensions of Coil.

$$A = \pi n a^2\left(1 + \frac{1}{3}\frac{d^2}{a^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where a denotes the mean radius, $2d$ the radial dimension of the section, and n the total number of turns. This formula may be proved thus. Since the number of layers in each coil was 12

Calculation of Effective Area (A) of Coil.

$$A = \frac{1}{12} n\pi\left\{\left(a - \frac{11d}{12}\right)^2 + \left(a - \frac{9d}{12}\right)^2 + \ . \ . \ + \left(a + \frac{11d}{12}\right)^2\right\}$$

$$= n\pi a^2\left(1 + \frac{d^2}{3a^2}\right)$$

nearly.

The value of the galvanometer constant G was calculated by an equation equivalent to that obtained from (9), p. 248 above, by taking the first term $2\pi\gamma a^2/r^3$, putting $\gamma = 1$, multiplying by n, and substituting for a^2/r^3, on account of the axial breadth $2b$ and radial depth $2d$, of the sections the value given in (20), p. 257, that is from

Calculation of Galvanometer Constant (G) of Coil.

$$G = 2\pi n\left\{\frac{a^2}{r^3} + \frac{b^2}{6}\frac{3a^2}{r^7}(4x^2 - a^2)\right.$$

$$\left. + \frac{d^2}{6}\frac{1}{r^7}(2x^4 - 11x^2a^2 + 2a^4)\right\} \quad . \quad (28)$$

where x = distance of mean plane of either coil from the suspension fibre.

The value of GA obtained after applying all corrections, and including in it allowances for non-verticality of the axis and for torsion of the fibre, was 29887600.

Value of GA.

The value of L was found by calculating the inductance for a coil of mean radius a and rectangular cross-section of which the length of diagonal was r. This was found from the formula

Calculation of Self-Induction of Coil.

$$L = 4\pi n^2 a\left\{\log\frac{8a}{r} + \frac{1}{12} - \frac{4}{3}\left(\theta - \frac{1}{4}\pi\right)\cot 2\theta - \frac{1}{3}\pi\,\mathrm{cosec}\,2\theta\right.$$

$$\left. - \frac{1}{6}\cot^2\theta\log\cos\theta - \frac{1}{6}\tan^2\theta\log\sin\theta\right\} \quad . \quad (29)$$

which is simply the formula (119′), p. 308, with the value of the logarithm of the geometric mean distance of the cross-section

from itself, given by (114) p. 302, put for log R. The dimensions of the coil used were those given by the B.A. Committee, viz. $a = 15{\cdot}8194$ cms., axial breadth of each coil 1·841 cm., radial depth 1·608 cm., and distance of mean planes apart 3·851 cms.

Calculation of Self-Inductance of Coil.

The inductance was computed for the double coil by adding together the self-inductances of the coils taken separately, and twice the mutual inductance of the two coils. For if L_1, L_2, be the self-inductances, and M their mutual inductance, the whole electrokinetic energy of a current γ is $\frac{1}{2}\gamma^2(L_1 + L_2 + 2M) = \frac{1}{2}\gamma^2 L$ if L be the self-inductance of the whole system. To the approximation given by (29) Lord Rayleigh found for $L_1 + L_2$ 30192000 cms., and for $2M$ 14582000 cms. Corrections for the finite size of the cross section, and (since the introduction of the geometric mean distance is made on the supposition that the coil may be regarded as straight) for curvature were made. The latter can be calculated by the series (156), p. 134, or by the elliptic integral formula by dividing the coil up into concentric circular filaments, and integrating over the cross-section. Lord Rayleigh found that for a single coil of circular cross-section of radius ρ the value of L is given by the equation

$$L = 4\pi n^2 a\left\{\log\frac{8a}{\rho} - \frac{7}{4} + \frac{\rho^2}{8a^2}\left(\log\frac{8a}{\rho} + \frac{1}{3}\right)\right\} \quad . \quad . \quad (30)$$

so that the correction for curvature increases L. The correction term for curvature in the case of a coil of the same mean radius a and square cross-section of the same area is very nearly the same as in this formula. It is thus an addition to the approximate value given by the equation (29) above. The corrections in L_1 and L_2 were each 11950 cms., and the correction on $2M$ 346900 cms., so that finally

$$L = 45141800 \text{ cms.}$$

The value of $2M$ found by the formula of quadratures given on p. 403, from the value given by the elliptic integral formula for two circles, was 14939400 cms., agreeing very closely with the value 14928900 cms. (14582000 + 346900) cms. already obtained.

Experimental Determination of Inductance

An experimental determination of L was made by the method described above, and gave 45000000 cms. on the supposition that the B.A. unit was 1 per cent. less than the ohm. The value given by Maxwell * uncorrected for curvature is 43744000,

* "On a Dynamical Theory of the Electromagnetic Field," *Phil. Trans. R.S.*, Vol. CLV. (1864), and Reprint of Papers, Vol. I. p. 596.

and the allowance for curvature, 734500 cms., is subtracted from instead of added to this value, giving finally with a correction for the finite diameters of the wires and variation of the current over the cross-section $L = 43016500$ cms. It is suggested by Lord Rayleigh that the discrepance may be due mainly to an interchange of the breadth and depth of the coils, together with the mistake just noticed as to the correction for curvature.

Observations.

The observations included (1) the resistance of the experimental coil as compared with a standard coil of German silver of nearly the same resistance, viz. 4·6 ohms, (2) the deflections produced by the spinning of the coil, (3) the speed of rotation.

Comparison of Resistances.

The comparison of resistances was made by a balance arranged by Mr. J. A. Fleming, in which Prof. Carey Foster's method (see Vol. I. p. 347) of interchanging the resistance to be compared with the standard was used to give the difference between the two resistances in terms of a certain length of the bridge wire. Error due to thermo-electric currents was eliminated by making the comparison with the battery current first in one direction, then in the other. A comparison was made at the beginning and end of each set of spinnings.

Suspended Needle. Form of Needle Used.

The needle consisted of four magnetized needles, each 15 cms. long, mounted on four parallel edges of a small cube of cork, to which the mirror was also fixed. This arrangement was adopted as four equal, thin, uniformly magnetized magnets placed along the parallel edges of a cube of length of edge $1/\sqrt{3}$ of the length of the magnet, form a needle the action of which is to a high degree of approximation the same as that of an infinitely small needle at the centre of the cube. The magnets were made about 2·3 times the edge of the cube in length to allow for non-uniformity of magnetization.

Adjustment of Needle.

The needle was adjusted in position by raising or lowering the cube until it was midway between the highest and lowest points of the circular frame, and then adjusting it in the two other directions, by attaching a pointer to the frame reaching in nearly to the centre, then turning the plane round, and observing whether the pointer occupied the centre of the small circle described by the point.

Determination of Ratio M/H for Needle.

The needle was in the usual manner caused to deflect another horizontally suspended needle in order to determine the ratio M/H of the magnetic moment to the horizontal component of the earth's magnetic field. At a distance of one foot the suspended needle was deflected through $\tan^{-1}\cdot000298$, and hence at a distance equal to the mean radius of the coil, 15·85 cms., the deflection of the needle would have been ·0021 approximately.

Now if r denote the mean radius of the coil, and μ the deflection of the needle, we have by (1), p 73 above, since the length of the magnet was small compared with r

$$\tan\mu = \frac{2M}{Hr^3}$$

and approximately $G = 2\pi n\,r$, and $A = n\pi r^2$, where n is the number of turns. Thus $r^3 = 2A/G$ and $\tan\mu = GM/HA$. This was used as the value of GM/HA in the term in (24) in which that quantity occurs.

Arrangement of Telescope and Scale. The telescope and scale (which was straight) were adjusted in the usual manner (see Vol. I, p 216). The distance of the scale from the mirror was compared with the scale directly, so that the absolute length of a scale division did not enter in the result. The following were the numbers

Distance of scale from mirror . .	252·28 cms
Correction for glass plate 3·2 mms. thick through which mirror was viewed $3{\cdot}2\left(1 - \frac{1}{\mu}\right)$	0·11 cms.
Distance (reduced)	252·17 cms.

The heights of the centre of the mirror and the centre of the objective above the line of the scale divisions were measured by means of a cathetometer, to obtain the data necessary for finding the inclination of the mirror to the horizontal. For this a correction was applied to the readings.

Correction for Torsion. The torsion of the silk fibre, which was 4 feet long, was also estimated by turning the magnet through 5 complete turns, and observing the deflection of the magnet. It was found that the magnet was shifted 5·6 divisions per turn, or through an angle of ·001107. Opposite turning of the magnet gave ·001117, so that the correction for torsion was obtained by calculating $\tau = {\cdot}00111/2\pi$, and using for A the value $A(1 + \tau)$.

Correction for Level. A correction for level of the coil was also applied, as it was found that the upper end of the axis was inclined towards the north by an angle ·0003 radian. The component of force at right angles to the axis was thus, if I be the intensity of the field, and D the dip, $I\cos(D - {\cdot}0003) = H(1 + {\cdot}0003\tan D)$ nearly. Thus for A was used finally the value

$$A(1 + {\cdot}0003\tan D)(1 + \tau$$

Mode of Driving Coil.

The spins were taken in sets of four at each speed. The coil was driven by a long cord from a water motor acting by the impulse of water on metal cups. To insure a constant pressure the motor was driven by water from a small cistern, which gave a head of 50 feet. The regulation of the motor was effected by observing that the work done by the motor is proportional to the difference between the speed of the jet and that of the cups, and to the speed of the cups. For, if the water is just reduced to rest the momentum of unit mass of water destroyed is v, the speed of the jet, and the mass of water received per unit of time is $a(v - v_1)$ if v_1 be the speed of the cups, and a the area of the jet. Thus the rate at which momentum is given by the jet to the cups is $av(v - v_1)$. The rate at which the motor works is therefore $av(v - v_1)v_1$. Thus at zero speed, and at the speed of the jet the water motor does no work. At half the latter speed the motor does work at the maximum rate. Thus the diagram of activity is a parabola with vertex upwards if speeds of the motor be taken as abscissae.

Action of Motor.

Possible Uniform Speeds of Driving.

Drawing on this diagram the curve of work done against resistances, we obtain from the points of intersection of the two curves the possible uniform speeds of running, and these speeds are more sharply defined the more nearly the curves are at right angles. Now the activity spent in overcoming resistance to the motion of the coil is a function of the speed v_1 of the form $Av_1 + Bv_1^2 + Cv_1^3 + \&c$ since there are included constant or frictional resistances, which give the first term, resistances such as viscous resistances which are proportional to the speed, which give rise to the second term, and resistances which vary as higher powers of the speed, such as resistance due to air set in motion by the cups, &c.

The curve of activity against resistance is therefore convex downwards, and at high speeds in the experiments there is no difficulty in obtaining definite enough intersection, but at low speeds this is not the case. It was necessary therefore at low speeds of the coil to run the motor fast, and use a reducing pulley in order to enable the curve of resistances to intersect at a suitable place.

Observation of Speed.

The speed of rotation was observed by the stroboscopic method, in which a card marked with circles of alternately black and white spaces (or "teeth") is viewed through narrow slits in thin plates of metal attached in the plane of vibration to the prongs of a tuning-fork. The slits overlap when the fork is at rest so that to an observer looking through them the card is visible; when the fork is in vibration vision is possible through the slits twice only in every complete vibration. (See Fig. 144.)

By Stroboscopic Card.

The fork was electrically maintained, and had a frequency of about 63½ (more nearly 63·69). Thus the card could be seen 127 times a second through the slits. Hence if a circle on the card marked with alternate black and white teeth passed the mean position of the slits a number of times a second equal to twice the frequency of the fork, the circle appeared to be at rest.

The card was graduated with five circles containing 60, 32, 24, 20, 16 black teeth respectively, to enable a variety of speeds to be observed without any change in the frequency of the fork. By looking over one end of one of the vibrating plates the card could be seen only once in each complete vibration, and thus the 60 teeth circle could be used for the lower speeds.

The contacts of the fork were made and broken with a platinum point and mercury cup covered with pure alcohol. The arrangement worked exceedingly well, and went for hours without requiring the smallest attention. A comparison was made, by means of beats, between the pitch of the fork and that of a standard fork.

It was found that the speed of the disk could be regulated by the observer by applying slight friction to the driving cord, when the teeth showed any tendency to pass. He therefore allowed the cord to run lightly through his fingers, and after a little practice it was possible so perfectly to regulate the speed that a tooth never passed the pointer except perhaps by inadvertence, when he at once brought it back by slightly retarding the cord. The passage of one tooth in each second meant of course only a variation of 1 in 127 in the speed.

Observation of Changes of Magnetic Declination.

In the course of the observations note was taken of the changes of magnetic declination by means of an auxiliary magnetometer set up near enough the revolving coil to be practically in the same magnetic field with it, but at the same time so far away as to be unaffected by the induced currents produced by the spinning. The scale was read by means of a telescope, and the distance from mirror to scale, 2½ metres, was the same as that of the mirror of the magnet in the coil from its scale, so that the corrections could be made by simple comparison of readings.

Effect of Air Currents in Magnet Box.

Some trouble was caused by air currents in the box containing the magnet; these currents caused change of zero during a set of spinnings. They were mainly due to radiation of the lamp and gas jets, and precautions were taken to diminish the effect by covering the magnet box with gold-leaf to reflect the heat as much as possible. The error from this cause, however, was not greater than that which necessarily affected the determinations

of the mean radius of the coil, and the distance of the mirror from the scale.

If ϕ be the deflection of the mirror, d the observed reading, and D the distance of the mirror from the scale, δ the distance of the zero of the scale from the centre, then, approximately

$$2D \tan \phi = d - (d - \delta)\frac{(d+\delta)^2}{4D^2} + \frac{d^5}{8D^4} \quad . \quad . \quad (31)$$

This formula was used for calculating $\tan \phi$, δ being taken positive when in the same direction as d. Irregularities in the scale were allowed for, and, as stated above, a correction applied for the slight non-horizontality of the normal to the mirror. The vertical distance between the centre of the objective and the point in which the normal intersected the scale being denoted by p, the angle between the normal and the horizontal by α, the correction was $dp\alpha/D$, which amounted to $d \times \cdot 00014$.

The resistance comparisons generally showed a rise of resistance during each set of experiments. This was corrected for on the supposition that the rise of temperature was uniform during the time elapsing between two successive measurements of resistance. The error arising from uncertainty of temperature did not amount to more than ·05 per cent.

The following is one set of readings in which C denotes the resistance of the coil, S the resistance of the standard. Specimen Set of Readings.

Time.		
9 h. 17 m.	Resistance Compared	$C = S + \cdot 0225$
9 h. 32 m.	Reading of Auxiliary Magnetometer	26·9
	Position of rest of needle	766·48

Time	Direction of Rotation.	Deflected Reading	Auxiliary Magnetometer Reading.
9 h. 37 m.	Negative	367·60	27·55
9 h. 42 m.	Positive	1166·40	28·24
9 h. 47 m.	Negative	366·23	28·50
9 h. 53 m.	Positive	1166·09	28·30

Time.		
9 h. 57 m.	Reading of Auxiliary Magnetometer.	27·2
	Position of rest of needle	767·08
10 h. 0 m.	Resistance compared	$C = S + {\cdot}0272$

From these the following table of corrected readings and deflections was found :—

Position of Rest.	Deflection observed.	Deflection corrected for Scale Errors and Temperature.
766·28	− 398·61	− 396·55
765·59	+ 400·81	+ 397·93
765·33	− 399·10	− 397·23
765·53	+ 400·56	+ 397·23
		Mean 397·42

$$C = S + {\cdot}0248.$$

Calculation of Value of Resistance.

The value of R was calculated directly from the solution of the quadratic (24) above. If A' be put for $A(1 + {\cdot}0003 \tan D)/(1 + \tau)$, the value of the area of the coil when it is made to include the correction for torsion and level, and $\tan\theta$ denote GM/HA as determined above (p. 568), this solution may be written

$$R = n\pi GA' \cot\phi \,\{1 + \tan\mu \sec\phi + \sqrt{(1 + \tan\mu \sec\phi)^2 - U \tan^2\phi}\} \quad . \quad . \quad (32)$$

where $U = (2L/GA')/(2L/GA' - 1)$, and n denotes the number of turns of the coil per second = 2 frequency of fork / number of teeth in stationary circle.

Table of Results.

The following table gives the result of all the experiments. Column 1 gives the date of the experiment, 2 the speed in terms of the number of teeth on the apparently stationary circle, 3 the deflection corrected for scale errors and variation of temperature during the set of experiments, 4 the absolute resistance of the revolving coil on the assumption that the inductance of

the coil was 4·5 × 10^7 cms., and S the absolute resistance of the standard German silver coil at 11°·5 C. as given by the different experiments, subject to a correction for the copper rods connecting the rotating coil with the resistance bridge.

Date	Teeth on Card	Deflection	$R \times 10^{-9}$	$R \times 10^{-9}$ for Standard Coil	Mean
Dec. 7	120	110·42	4·5486	4·5419	4·5364
10		110·22	4·5568	4·5309	
Dec. 2	60	218·61	4·5580	4·5487	4·5467
6		218·30	4·5620	4·5471	
10		218·72	4·5531	4·5422	
Dec. 2	32	397·75	4·5639	4·5417	4·5427
6		397·39	4·5672	4·5415	
10		397·26	4·5687	4·5448	
Dec. 2	24	513·73	4·5719	4·5446	4·5442
6		513·58	4·5734	4·5438	

Mean R = 4·5427 × 10^9, in cms. per sec.

Final Mean Value of Resistance of Coil-Circuit. Value of B.A. Unit.

The value of L here used was slightly less than that found by Lord Rayleigh, and agreed very closely with a value (4·5130 × 10^7 cms.) deduced by the method of least squares from the results for different speeds.

The German silver standard was then compared with the original standards prepared by the B.A. Committee. The standard was found to be 4·595 B.A. units at 11°·5 C., and the resistance of the copper rods connecting the rotating coil with the bridge was found to be ·003 unit. Thus 4·592 B.A. units were found to be equivalent to 4·5427 × 10^9 in cms. per second, or

$$1 \text{ B.A. unit} = \cdot 9893 \times 10^9, \text{ in cms. per second.}$$

Lord Rayleigh's Further Experiments.

The investigation just described was repeated by Lord Rayleigh with improved apparatus. The coil was made more massive to remove risk of deformation by the winding, and its

Construction of Ring. dimensions were increased in the ratio of about 3 to 2. The ring was in two halves, joined along the horizontal diameter by projecting flanges, and insulated from one another by a layer of ebonite. Its construction with driving arrangements, &c., is shown in Fig. 129.

Ring Currents. The ring having been wound was spun with its circuit open, and it was found that a perceptible effect on the magnet was produced. This was traced to currents circulating in the parts of the ring adjacent to the ebonite layer, where there was

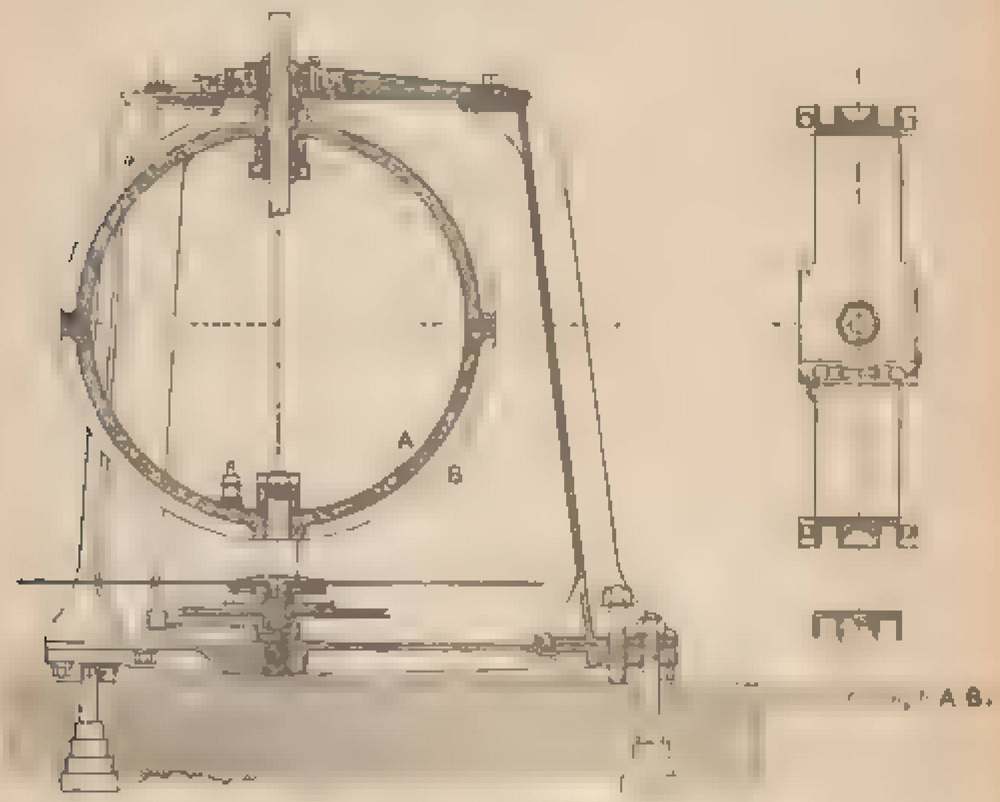


FIG. 129

sufficient body of metal to give currents in circuits at right angles to the windings. These currents were afterwards allowed for.

Arrangement of Magnet. To obviate air disturbances of the needle caused by rotation of the coil, the magnet box was screwed air-tight to the lower end of a brass tube which passed through the upper part of the axis of rotation. By unscrewing the box and pulling up the brass tube the magnet could be withdrawn with the fibre intact. The level of the needle was adjustable by means of a sliding piece,

to which the upper end of the fibre was attached. The whole arrangement was so rigid that no disturbance was produced by the air even at the highest speeds.

The needle was on the same plan as before. Its moment was however six or seven times as great, with, on account of the greater dimensions of the coil, a value of ·0042 for MG/AH, ($\tan\mu$), or only about twice the former value. (This was determined in a manner similar to that already described). The horizontal breadth of the mirror was diminished, and thus with greater magnetic moment, and smaller mirror the disturbance from air currents inside the box was brought down to about 1/15 of what

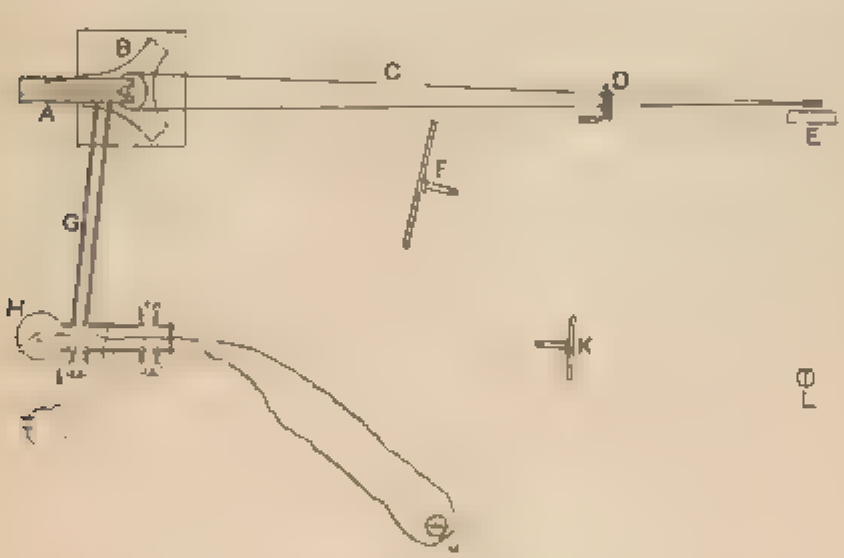


FIG. 130

A. Stand for suspended parts.
B. Frame of revolving coil.
C. Driving cord.
D. Fork and Telescope.
E. Water motor.
F. Telescope and scale for observing deflections.
G. Copper bars connecting to bridge.
H. Fleming's bridge.
I. Platinum silver standard.
J. Bridge galvanometer.
K. Telescope and scale of auxiliary magnetometer.
L. Auxiliary magnetometer.

it was in the former apparatus. The period of oscillation was brought up to a convenient amount by an inertia ring ¾ inch in diameter added to the magnet. The weight of the whole was so small that it was easily borne by a single fibre of silk.

The coil was driven and its speed determined as in the former experiments.

The resistance of the coil being 23 units as compared with the former value 4·6 units, arrangements were made to add resistances to the copper circuit when the variation of resistance passed beyond the range of the slide wire, and a platinum-silver standard of about 24 units was employed.

The general arrangement of the apparatus is shown in Fig. 129.

A first set of spinnings gave less accurate results than were expected, and the cause was traced to the paper scales. These were then replaced by scales engraved on glass. Some trouble was also caused by an imperfect mercury contact at the junction of the copper coil with the bridge connections; but when this was remedied the arrangements worked satisfactorily.

Dimensions of Coil.

The dimensions of the coil were as follows :—

	Mean circumf. cms.	Mean radius. cms.	Axial breadth. cms.	Radial breadth. cms.
Coil *A*	148·53	23·639	1·99	1·59
Coil *B*	148·35	23·611	1·99	1·54
	Mean 148·44	Mean 23·625		

Each coil was wound with sixteen layers of eighteen windings in each layer, except the eleventh layer of *A*, which had seventeen turns. An extra turn was laid on *A* outside the sixteenth layer.

Each layer was measured during winding, and again on unwinding after the experiments had been made. Thus the effect of the pressure of the layers in diminishing their radii was estimated. The mean of the mean radii of the two coils was then 23·616. Weights of two to one were given to the last result and the former so that a mean of 23·619 cms. was adopted.

GA was calculated from the formulæ (27), (28), above, multiplied together, and it was found that $\log (GA) = 8·17682$. The correction for level and torsion, it was found, increased this number only to 8·17686.

The value of L for the coil was found by calculating L_1, L_2, and M for the two coils as explained above (p. 566), L_1, L_2 were

found by (29), and M by the formula of approximation given at p. 403 above. Thus

Calculation of Inductance.

$$L_1 \text{ (for } A) = 1029{\cdot}3 \times 16^2 \times 18^2 \text{ cms.,}$$
$$L_2 \text{ (for } B) = 1031{\cdot}9 \times 16^2 \times 18^2 \text{ cms.,}$$
$$2M = 832{\cdot}88 \times 16^2 \times 18^2 \text{ cms.,}$$

so that

$$L = L_1 + L_2 + 2M = 2{\cdot}4004 \times 10^8 \text{ cms.,}$$

subject to a correction for curvature.

L was also determined experimentally. A full account of the determination is given in Chap. IX. above. The final result thus found was $L = 2{\cdot}4052 \times 10^8$ cms.

The currents in the ring were allowed for as follows. Putting $\tan\mu$ for $MG\ AH$ as at p. 568 above, and A', G', L', R, for the quantities depending on the ring and corresponding to A, G, L, R, we have from (24)

$$\tan\phi + \tau\frac{\phi}{\cos\phi} = \tfrac{1}{2}\frac{GA\omega}{R^2+\omega^2L^2}(R + L\omega\tan\phi + R\tan\mu\sec\phi)$$
$$+ \tfrac{1}{2}\frac{G'A'\omega}{R'^2+\omega^2L'^2}(R' + L'\omega\tan\phi + R'\tan\mu\sec\phi) \quad . \quad (33)$$

if the wire circuit is closed. If the wire circuit is open and the speed is the same

Theory Ring Currents

$$\tan\phi_0 + \tau\frac{\phi_0}{\cos\phi_0} = \tfrac{1}{2}\frac{G'A'\omega}{R'^2+\omega^2L'^2}(R' + L'\omega\tan\phi_0 + R'\tan\theta\sec\phi_0) \quad (34)$$

Putting $\tau\tan\phi$ for $\tau\phi\cos\phi$, and $\tau\tan\phi_0$ for $\tau\phi_0/\cos\phi_0$, neglecting the terms multiplied by $R'\tan\theta$, and subtracting, we get after reduction

$$\tan\phi - \tan\phi_0 = \tfrac{1}{2}\frac{GA\omega}{(R^2+\omega^2L^2)(1+\tau)}(R + L\omega\tan\phi$$
$$+ R\tan\mu\sec\phi)\left(1 + \frac{L'\omega}{R'}\tan\phi_0\right) \quad . \quad (35)$$

Thus the effect of L' would be to increase the deflections at high speeds beyond their proper values, whereas that of L is to diminish them. The value of L R for the wire circuit was ·01 second : for the ring L'/R' was no doubt much less, and further $\omega\tan\phi_0$ at the highest speed was only 1/26. The last factor on the right of (35) may be omitted. Hence R is given by (32)

above with $\tan\phi - \tan\phi_0$ used instead of $\tan\phi$, (but $\sec\phi$ left unchanged) and

$$U = (2L/G_1A)\{2L/G_1A - \tan\phi/(\tan\phi - \tan\phi_0)\},$$

where G_1 denotes $G/(1+\tau)$.

Mode of Carrying out Observations. With regard to the observations, the general mode of carrying out the work and correcting the results was the same as in the former investigation. An auxiliary magnetometer was used as before to trace changes of declination, and the speed and deflections were read off as formerly. For the highest speed it was found that $\tan\phi_0/\tan\phi = 7\,81\,439{\cdot}41$, and this with the value of G_1A stated above gave $\log_{10} U$ 84325.

The standard coil was kept immersed in water the temperature of which was observed, and the temperatures of the air were also observed in the neighbourhood of the copper coil, and near the standard tuning fork by which the frequency of the speed-measuring fork was determined.

Comparisons of the resistance of the copper coil with the platinum-silver coil were made before and after each set of spinnings. The resistance of the copper circuit was equal to that of the standard coil + or − the resistance of the bridge wire required for balance.

Specimen Set of Readings. A specimen set of readings is here given with the necessary corrections. The first set of six were made with the wire circuit open, the second set with it closed.

	No. of spinning	Time	Magnet reading corrected by auxiliary magnetometer	Diff	Mean deflections
		H. M.			
Wire circuit open	1 −	8 16	593·38	10·48	
	2 +	8 18	603·86	10·45	
	3 −	8 20	593·41	10·69	5·29
	4 +	—	604·10	10·65	
	5 −	8 23	593·45	10·60	
	6 +	8 25	604·05		
Wire circuit closed	7 +	8 45	901·58	605·47	
	8 −	8 47	296·11	605·43	
	9 +	8 50	901·54	605·12	302·56
	10 −	8 52	296·42	604·91	
	11 +	8 55	901·33	604·77	
	12 −	8 58	296·56		

The resistance of the standard $-$ the resistance of the copper circuit expressed in terms of the resistance of one division of the bridge wire as unit, was 212 at the beginning of the second six observations, and $-316{\cdot}5$ at the end, giving a mean of -52 during the interval. But each division of the bridge wire was about 1/480000 of the whole resistance of twenty-four ohms, so that if balance had been obtained on the average at the middle of the bridge wire the deflection would have been 302·59.

Correction of Results.

Again the temperature of the standard during the experiments had a mean value of 10°·025, so that the resistance of the standard which for this series was taken as normal at 13°, was below its normal value, and the deflections were too large. The variation of resistance of the standard per degree was 3 parts in 10000, so that the deflection fell to be diminished by about 2·7 parts in 3000 or by ·27.

The standard number of beats per minute between the standard fork and the electrically-maintained fork (at 17° C.) was taken as 59 during the series of observations, and in the set of observations here taken as a specimen the number of beats was $56\frac{1}{2}$ per minute, so that the electrically-maintained fork was too sharp by $2\frac{1}{2}$ parts in 60×127, 127 being very nearly twice the frequency of the latter fork, that is the speed was too great by this amount. This gives as the correction of the deflection for excess of speed $-{\cdot}10$.

But the standard fork which was at normal frequency at 17° was at 13°·05, and therefore vibrated more quickly than the normal rate. The amount of quickening was about 1 in 10000 per degree of difference of temperature. Thus there was a further temperature correction on the deflection of $-{\cdot}12$.

Adding together and applying these three negative corrections, we get for the deflection which would have been obtained if everything had been in its normal state as specified 302·10.

Corrected Value of Mean Deflection. Results of Experiments.

From the series of experiments made at different speeds, it was seen that there was a tendency for the value of the resistance to rise with the speed. This would have been the effect of an under-estimate of the value of L, but as the error to account for the discrepancies at the different speeds would have had to be about 1 per cent., it was taken as more probable that there were ring currents generated which were not conjugate to those in the wire circuit. There was no doubt, however, that the true value would be obtained, no matter which of these views was taken, by applying a correction proportional to the square of the speed. This correction was calculated from two extreme speeds and applied to the results. Thus the principal series of experiments consisting of many different sets of spinnings gave the

numbers in the following table as their final corrected result :—

Speed in teeth on card	Uncorrected resistance of standard at 13° (unit 10^9 C.G.S.).	Correction proportional to square of speed.	Corrected Resistance of standard (unit 10^9 C.G.S.).
60	23·619	·006	23·613
45	23·621	·011	23·610
35	23·630	·018	23·612
30	23·638	·025	23·613
	Mean 23·627		Mean 23·612

The result of this set of experiments was taken as that with which the B.A. standards should be compared. Another series made, however, gave practically the same result, viz. $23·618 \times 10^9$ C.G.S. units as the resistance of the standard coil at 13°.

Value of B. A. Unit. A careful comparison of the resistance of the standard coil with the B.A. unit gave

$$23·612 \times 10^9 \text{ C.G.S. units of resistance} = 23·9348 \text{ B.A. units,}$$

or $$1 \text{ B.A. unit} = ·98651 \times 10^9 \text{ C.G.S. units.}$$

Method of Lorenz. Lord Rayleigh and Mrs. Sidgwick have made a very careful determination of the value of the B.A. unit of resistance by the method of Lorenz. A disk of metal touched near its centre and at its circumference by the terminals of a conductor was spun round its axis of figure at a uniform observed speed, in the magnetic field of a coaxial coil carrying a current. The electromotive force produced in the circuit thus formed was balanced by the difference of potential between the terminals of a resistance through which flowed the current, or a known fraction of the current producing the magnetic field.

Theory of Method.

Supposing the disk touched at its centre, the total change in the flux of induction through the circuit in one turn is equal to the induction produced by the coil through the circular edge of the disk, or if M denote the mutual inductance of the coil and this circle, and γ the current, it is $M\gamma$. If n revolutions of the disk be made per second the electromotive force is $nM\gamma$. This is balanced by the difference of potential $R\gamma$ between the terminals of a conductor of resistance, R, and so we have

$$R = nM \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

M is calculated from the known data of the coil and thus R is found.

Lord Rayleigh and Mrs. Sidgwick's Experiments.

In no practical case can nM be large, and therefore R must be small, and a difficulty arises on this account in the carrying out of the method. This was overcome in Lord Rayleigh and Mrs. Sidgwick's experiments by arranging that the main current should flow along AC (Fig. 131), through a resistance a small

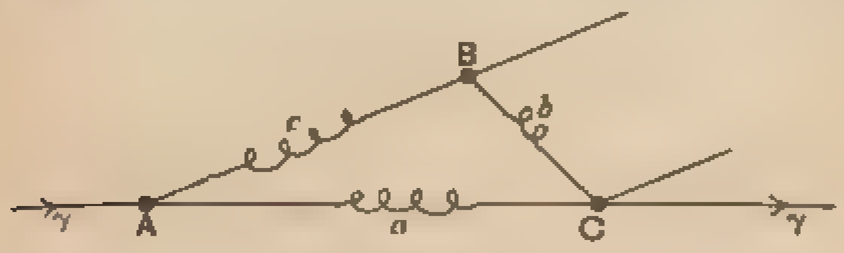


Fig. 131.

compared with the resistance c between A and B, while at the two points B and C, including a resistance b also small compared with a, the terminals connected with the revolving disk were applied. Thus b was the resistance which was evaluated by the experiment. The connections at A, B, C were made by means of mercury cups.

The main current being γ, and no current flowing in the circuit applied at BC, the current through $A\ B\ C$ is $\gamma a/(a+b+c)$. Hence the difference of potential between B and C was $\gamma ab/(a+b+c)$. This was therefore the electromotive force

generated by the motion of the disk. It will be convenient to regard it as the difference of potential produced by the current γ between the ends of a conductor of resistance $ab/(a+b+c)$.

Arrangement of Apparatus. The pair of coils used by Glazebrook in his determination of the ohm (see above, p. 543) were employed, and were at first placed close together with the disk between them, so as to give a maximum inductive effect. The axle was mounted vertically in the frame already used for the spinning coil determinations, so that the arrangements then used for driving and measuring the speed were available also in the present case.

The diameter of the disk was about ·6 of that of the coils. This size was chosen as on the one hand it was not desirable to

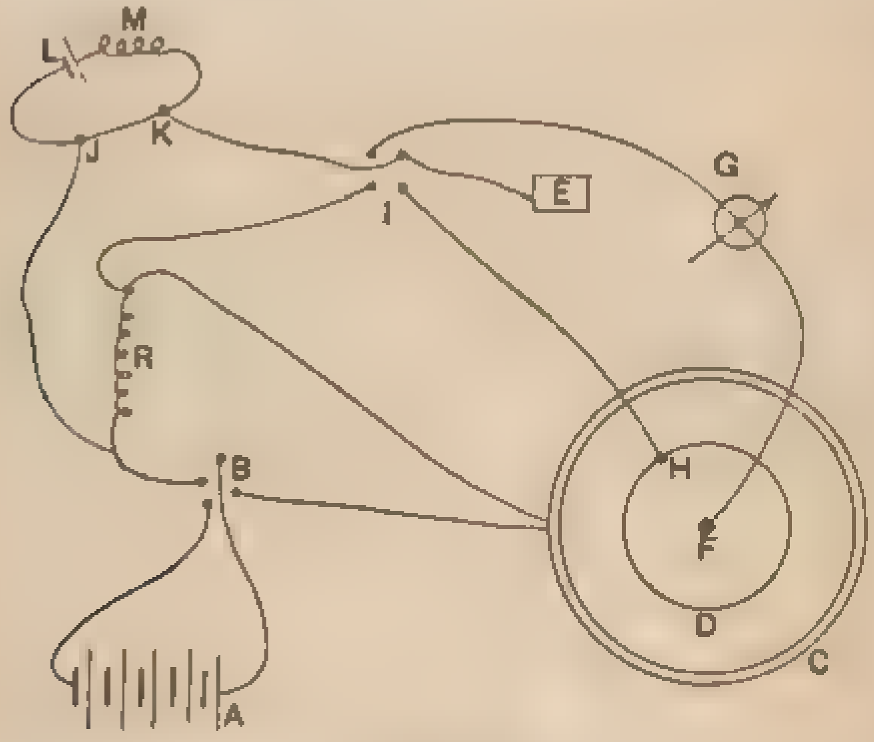


FIG. 132.

have any part of the disk near the wire, on account of the more rapid variation there of the magnetic induction, and the consequent greater importance of errors in the estimation of the radius of the coils or disk, and on the other hand too small a radius rendered the arrangement insensitive.

After some trials it was decided to make the edge cylindrical, and to make the edge contact by a brush of fine copper wires placed tangentially to the edge and amalgamated with mercury.

The arrangement of the apparatus is shown diagrammatically in Fig. 132. The battery A is connected with a mercury cup commutator B, by which the current can be sent in either direction through R. R is here taken as a simple conductor, but the shunt arrangement was of course used, and R may be taken as standing for the resistance $ab/(a+b+c)$.

The terminals F and H attached to the centre and circumference of the disk were connected with a mercury reversing key I, and in one of them was included a reflecting galvanometer G. From I the wires of the disk circuit proceeded to the terminals of R, one of them however having included in it a portion, JK, of a circuit containing a sawdust Daniell L, and a resistance coil of 100 ohms.

The latter circuit was designed to balance the effect of thermo-electric force at the sliding contacts of the brush on the disk, and the inductive effect of the earth's magnetic field in which the disk rotated, which would have given a current through the sliding contacts, thereby bringing these resistances into the account. The function of the galvanometer G was to test this balance, and that required when the disk was rotated.

The battery and frame carrying the disk were insulated from the ground, and the coils insulated by ebonite supports, and for definiteness one point of the galvanometer was connected to earth at E. It was found that there was no error from leakage.

Mode of Carrying out Experiments.

In the carrying out of the experiments the test of perfect balance of the electromotive force of the disk, together with the thermo-electric force and inductive action of the earth's field, above referred to, was absence of deflection of the galvanometer needle when the battery current was reversed. It was not however thought desirable to seek accurate balance, but to make observations of the effect on the galvanometer reading of reversal of the battery current with a resistance R_1, very little different from that (R) needed for balance. After a series of readings had been taken, R_1 was changed to R_2, which was such that the same reversal of the current was accompanied by a galvanometer deflection of opposite sign to the former. The two series of results gave R by interpolation.

To eliminate progressive change in the battery electromotive force, the observations for R_1 were interspersed with those for R_2. As soon as each series of results had been obtained for one direction of driving, the driving cord was reversed and a similar series of observations made.

Preliminary trials proved that the shunt arrangement represented in Fig 131 was faulty. The pieces dipping into the cup

Shunt Arrangement for Resistance balancing E.M.F. of Disk.

C were moved from day to day to verify the contacts, and the fact was overlooked that as the main current also traversed C, a small change in the positions of the contacts might make a considerable difference. For any uncertainty, even of very small absolute amount, would affect both a and b, which were small, and therefore seriously $ab\ (a + b + c)$.

The arrangement shown in Fig 133 was accordingly adopted. Two cups, A, D, were connected by two 1 unit coils, through which the main current flowed, while two other mercury cups, B, C, received the galvanometer terminals of the disk circuit. C was connected with D by a stout rod of copper. A resistance box E was placed as a shunt across A to enable the resistance of the shunt to be adjusted.

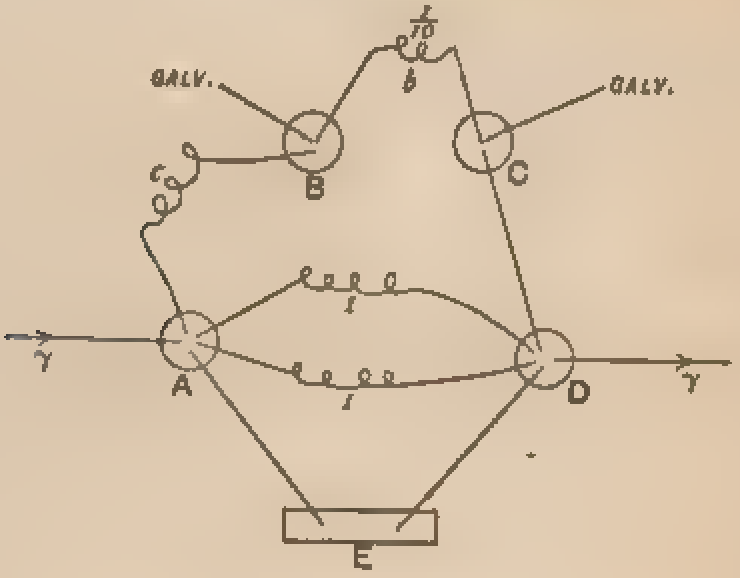


FIG. 133.

Arrangement of Coils to diminish Effect of Error in Mean Radius.

Two series of results were taken with the coils close together, and a third series with the coils separated to a position in which the disk, midway between them, was so situated that the induction through it was as nearly as possible independent of variations of the mean radius of the coils. That there was such a position is clear from the fact that, for given values of the radius of the disk and the distance of the plane of the disk from the mean plane of either coil, the induction is zero, both when the mean radius of the coil is 0 and when it is infinite. Hence there was some value of the mean radius of the coils for which the

induction was a maximum, and at which therefore the rate of variation of M with change of mean radius was zero.

For this purpose the coils were separated by distance-pieces of proper size; and to eliminate uncertainty as to the position of the mean planes relatively to the bobbins, after one set of observations had been completed, the bobbins were reversed on the distance-pieces, and another set of observations taken.

Dimensions and Distance apart of Coils.

The dimensions of the coils are given above (p. 543), and the distance of their mean planes apart in the close position was 3·275 cms. In the separated positions the distances apart of the mean planes were 30·681 cms. and 30·710 cms. respectively.

Dimensions of Disk.

The diameter of the disk was measured by callipers, and its circumference by a steel tape. It was found that the edge was slightly conical, and it was estimated that the mean diameter at the contact of the brush was 31·072 cms. The other contact was made at the shaft, and the diameter of the circle of contact there was 2·096 cms.

Calculation of Mutual Inductance of Coils and Disk.

The coefficient of mutual induction was calculated first by the elliptic integral formula (by aid of the tables given in the Appendix) for two circles of radius equal to the mean radii of either coil and disk, and at a distance apart equal to the distance of the mean plane of the coil from that of the disk. Then the cross-section of the coil was taken into account by the formula of quadratures given above (p. 403).

If a, a', be put for the radii of the coils and disk respectively, and x for the distance apart of the mean plane of the coil and of contact on the disk, $2b$ and $2d$ the axial breadth and radial depth of the coils, and $M(a, a', x)$ the result for the two circles, the results in cms. per turn of wire were as follows:—

Coils near together.

$a = 25{\cdot}760$ cms. $\quad a' = 15{\cdot}536$ cms. $\quad x = 1{\cdot}637$ cm.
$b = \ \ {\cdot}948$ cm. $\quad d = \ \ {\cdot}955$ cm.

$$
\begin{aligned}
M(a, a', x) &= 215{\cdot}4674 \\
M(a + d, a', x) &= 205{\cdot}1917 \\
M(a - d, a', x) &= 226{\cdot}9835 \\
M(a, a', x + b) &= 211{\cdot}7246 \\
M(a, a', x - b) &= 217{\cdot}5972.
\end{aligned}
$$

Adding to twice the first of these values the sum of the others,

and taking $\frac{1}{4}$ of the result, the average value of M for one turn of wire was given by

$$M = 215{\cdot}405.$$

When the coils were separated by the insertion of distance-pieces, so that $x = 15{\cdot}3472$ cms., without change of the other data, the corresponding values found were

$$\begin{aligned}
M(a, a', x) &= 110{\cdot}9240\\
M(a + d, a', x) &= 111{\cdot}2573\\
M(a - d, a', x) &= 110{\cdot}2442\\
M(a, a', x + b) &= 104{\cdot}5571\\
M(a, a', x - b &= 117{\cdot}6579
\end{aligned}$$

which gave (again for one turn)

$$M = 110{\cdot}926.$$

Effect of Errors in Measurement of Coils.

The effect of errors in the measurement of a, a', and x can be estimated by the formula

$$dM = \frac{\partial M}{\partial a}da + \frac{\partial M}{\partial a'}da' + \frac{\partial M}{\partial x}dx,$$

conjoined with

$$\frac{a}{M}\frac{\partial M}{\partial a} + \frac{a'}{M}\frac{\partial M}{\partial a'} + \frac{x}{M}\frac{\partial M}{\partial x} = 1,$$

which holds because the expression for M is homogeneous in a, a', x. Writing the last equation in the form

$$\lambda + \mu + \nu = 1,$$

we have for the first

$$\frac{dM}{M} = \lambda\frac{da}{a} + \mu\frac{da'}{a'} + \nu\frac{dx}{x}.$$

Now we may take it that approximately

$$\lambda = \frac{M(a + d, a', x) - M(a - d, a, x)}{2d}\,\frac{a}{M},$$

and similarly for μ, ν.

Thus for the case of the coils near together

$$\lambda = -1\cdot36, \quad \mu = -\cdot02, \quad \nu = 2\cdot38,$$

and for that of the separated coils

$$\lambda = \cdot123, \quad \mu = -\cdot956, \quad \nu = 1\cdot833.$$

Thus in the former case the importance of an error in the estimation of a is of rather more than half the importance of an equal proportional error in x, while an error in the estimation of a' is relatively unimportant. On the other hand, by the separation of the coils the importance of an error in a is diminished to about 1/11 of its former amount, while that of an error in a' is enhanced. The numbers show that the separation had been carried rather beyond its proper amount.

Deduction for Inductance over Circle of Internal Contact.

From the values of M in both cases had to be subtracted the part, M_0, say, corresponding to the small circle touched by the inner brush. The area of this circle was $\frac{1}{4}\pi \times 2\cdot096^2$; and therefore taking the magnetic force at the centre of the disk due to unit current in the coil of mean radius a as a sufficiently near approximation to the average induction over this circle, we get

$$M_0 = \frac{2\pi a^2}{(a^2+x^2)^{\frac{3}{2}}} \times \frac{1}{4}\pi \times 2\cdot096^2.$$

This was equal to $\cdot836$ in the first case, and to $\cdot534$ in the other.

The resistances, the arrangement of which is shown in Fig. 133, were the same in all three series of experiments. The coil b was of German silver and had a resistance of $\frac{1}{10}$ unit nearly, the resistance, a, between A and D was made up of two standard single units, and 7 or 8 B.A. units from the resistance box all in multiple arc.

Comparison of Absolute Resistance with B.A Unit.

In the first series of experiments c was a [10], in the second [10] + [5] + [1], and in the third series [10] + [5] + [5]. The resistances of the single units were already known, the others, that is the [10], [5], [5], [$\frac{1}{10}$], had to be carefully compared with standard B.A units. The [5]'s were compared by comparing first one of them with 5 units in series, and then the two [5]'s with one another; afterwards the sum of the two [5]'s was compared with the [10], the value of which was found by a special process.

Bridge Arrangement for Comparison of Small Resistance with Standard.

Three German silver coils of about 3 units each wound on the same tube, had their ends arranged so that they could by mercury cups be put either in multiple arc or in series, and a change made in a very small interval of time from one arrangement to the other. In multiple arc they were compared with a standard [1], and found to have a resistance $1 + a$. The arrangement was now rapidly changed to series, and the resistance became very nearly $9(1 + a)$. The standard unit was now added, and the resistance became $10 + 9a$. This was compared with the [10], the value of which was to be found. If there was a difference β, then $[10] = 10 + 9a + \beta$.

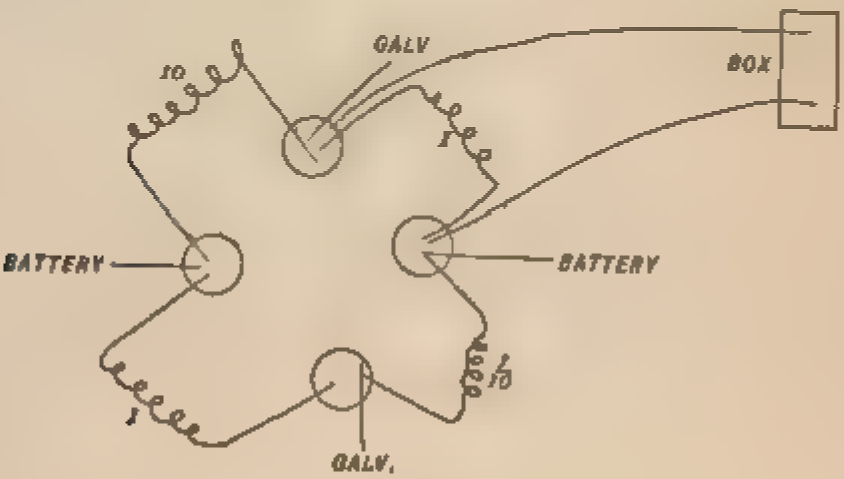


FIG. 134.

The [1/10] was determined as follows. Two standard units, the [10] and the [1/10], were joined as shown in Fig. 134 as a Wheatstone bridge, in which the battery and galvanometer terminals were, as shown, brought into direct contact with those of the [1/10] in the mercury cups. A resistance box containing coils up to 10000 was placed in multiple arc with one of the units to enable the latter to be adjusted to balance with all necessary accuracy. The four coils were so nearly in proportion that a resistance of several hundred units was required from the box to give balance, so that the delicacy of the arrangement was very great.

Specimen Set of Results.

As a specimen of the results showing the mode of applying the various corrections the table of results given for the second series of experiments with the coils near together is here reproduced.

COILS NEAR TOGETHER.

SPEED OF DISK ABOUT 8 REVOLUTIONS PER SECOND.

APPROXIMATE RESISTANCES $a = \frac{1}{2}$, $b = \frac{1}{10}$, $c = 16$.

Date	Effective resistance (B A units) used.	Difference of reading of galvanometer on reversal of current.		Effective resistance in B. A units corresponding to zero difference in galvanometer		Correction for change of speed of fork.				Effective resistance (B. A. units) as finally corrected.		Mean of effective resistance for both directions of rotation.
Aug 1882.		Rotation +	Rotation −	Rotation +	Rotation −	Beats between forks.	Correction to 72 beats.	Temperature of standard fork.	Correction to 16°.	Rotation +	Rotation −	
7th	·0027827 ·0028126	8·2 + 20·1	+ 9·4 − 21·5	·0027914	·0027918	73	− ·0000004	17·0	+ ·0000005	·0027915	·0027919	·00279170
8th	·0027821 ·0028120	− 8·6 + 21·0	+ 9·5 − 19·1	·0027908	·0027920	73	− ·0000004	17·2	+ ·0000004	·0027908	·0027920	·00279140
9th	·0027826 ·0028125	− 7·9 + 19·5	+ 7·5 − 19·1	·0027912	·0027910	72	0	17·6	+ ·0000005	·0027917	·0027915	·00279160
									Means	·00279180	·00279180	·00279157

Final Results of Experiments.

The first series gave $R = \cdot 00443407 \times 10^9$ B.A. units; hence the ratio of the B.A. unit to 10^9C.G.S. units of resistance being x, the absolute value of R was $x \times \cdot 00443407 \times 10^9$ C.G.S. But the value of M was M_1 multiplied by the number of turns in the coil (1588), and n the number of revolutions per second = 2 × frequency ÷ number of teeth stationary on card. Hence by (36) for the first series, since $n = 128\cdot 407/10$,

$$x \times \cdot 00443407 \times 10^9 = 12\cdot 8407 \times 214\cdot 569 \times 1588$$

or

$$x = \cdot 98674.$$

The second series gave, since for it $n = 129\cdot 340/16$ and $R = \cdot 00279157 \times 10^9$,

$$x = \frac{214\cdot 569 \times 1588 \times 129\cdot 340}{\cdot 00279157 \times 10^9 \times 16} = \cdot 98669.$$

In the third series $n = 129\cdot 340/10$, and $R = \cdot 00229762 \times 10^9$, so that from it

$$x = \frac{110\cdot 392 \times 1588 \times 129\cdot 340}{\cdot 00229762 \times 10^9 \times 10} = \cdot 98683.$$

Taking the mean of the first two results, and giving it the same weight as the last Lord Rayleigh found as the final result of the investigation,

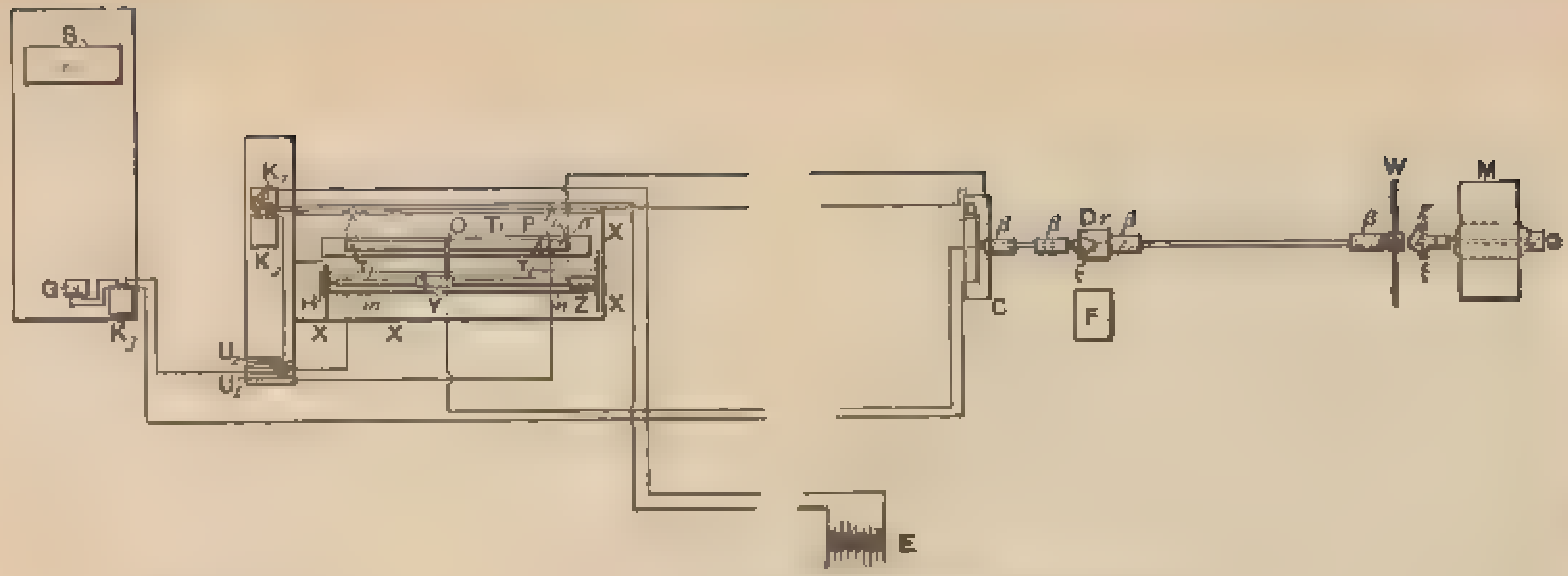
$$1 \text{ B.A. unit} = \cdot 98677 \times 10^9 \text{ C.G.S.}$$

With the value of the specific resistance of mercury in terms of the B.A. unit found by Lord Rayleigh and Mrs. Sidgwick, this gives 1 ohm = resistance at 0° C. of a column of mercury 106·214 cms. long and 1 sq. mm. in cross-section.

Absolute Determination of Sp. Resistance of Mercury.

A carefully planned and executed determination by Lorenz's method was made in 1891 by Prof. J. V. Jones, of Cardiff, who used in the construction of his apparatus the most accurate obtainable engineering appliances.

The standard coil consisted of a single layer of double silk-covered wire, ·02 inch in diameter, wound on a cylinder of brass about 10·5 inches in radius, in a screw thread of pitch ·025 inch. This cylinder was very carefully turned, and the screw thread cut on an accurate Whitworth lathe, and great care was taken to test the figure of the cylinder after it was finished. It was found that the cross section of the cylinder, instead of being circular, was always slightly oval, however many cuts were made over its surface, showing apparently an effect of internal stresses.



C. Standard coil.
D Disk
Dr. Stroboscopic cylinder.
B, β β Bearings
Tr Mercury trough.
T_1, T_2 Thermometers in mercury.
O Movable electrode.
P Fixed electrode.
E. Battery.

G, S. Galvanometer, camp, and scale.
M. Electromotor
ϵ Flexible coupling.
W. Fly wheel.
m, m Measuring machine.
Y, Z Loose and fixed headstock on measuring machine
K_1 K_2, K_3. Keys
XXXX. Box surrounding trough and measuring machine.
U_1, U_2 Adjusting coils.

FIG. 135.

Fixing of Mean Plane of Helix.

After the screw had been cut the mean plane of the coil was determined for the after placing of the disk in the following manner. The slide-rest of the lathe was made to carry a *V* tool, and a microscope, so adjusted that the image of the point of the tool was seen exactly at the centre of the graduated plate in the focal plane of the eyepiece. When the slide-rest was moved along the bed, the tool passed inside the cylinder while the microscope remained outside. The guide-screw of the slide-rest (of pitch ·25 inch) was turned by a wheel 9·75 inches in radius divided into 360 parts, and it was possible to estimate the position of the wheel to 1/10 of a division. By drawing, then, a generating line along the cylinder, and reading on this wheel the position of the microscope when the ridges of the first and second threads on this line were focussed in the field of view, then running the microscope along the generating line, and taking in like manner the readings for the last ridge and last ridge but one, the reading for the mean plane could be at once found. The mean of the first two readings subtracted from the mean of the last two gave obviously the distance between the first hollow and the last, and half the sum of these two means therefore gave the required reading. The tool was then moved to this position by the wheel and guide-screw, and a cut made round the inside of the cylinder at the plane thus found.

Winding of Coil.

At the intersection of the first and last hollows with this generating line small holes were bored radially through the brass of the cylinder, and were bushed with paraffined ebonite to receive the ends of the wire. The wire was secured at one end in the hole there, and was then laid on in the screw thread by the lathe, under uniform tension given by a weighted pulley. The ends of the wire were secured by melted paraffin run into the bushes, and blocks of ebonite attached to the cylinder at the ends of the generating line, on which the coil began and ended, carried binding screws, to which the ends of the wire were soldered.

Arrangement of Disk.

The arrangement of the apparatus is shown in Fig. 135. The disk was insulated from the axle by ebonite, and was fixed coaxially as described below in the mean plane of the coil. It was driven by an electromotor coupled direct, and was rotated in position and ground true by an emery wheel driven rapidly by an electromotor. Its diameter was measured by a Whitworth measuring machine. This consisted of a graduated bed carrying two headstocks, one fixed, the other movable, along the bed, by a guide-screw turned by a divided wheel. The distances used on the bed were compared with a standard scale.

A side view of the coil, disk, stroboscopic cylinder, &c. (for

Fig. 136.

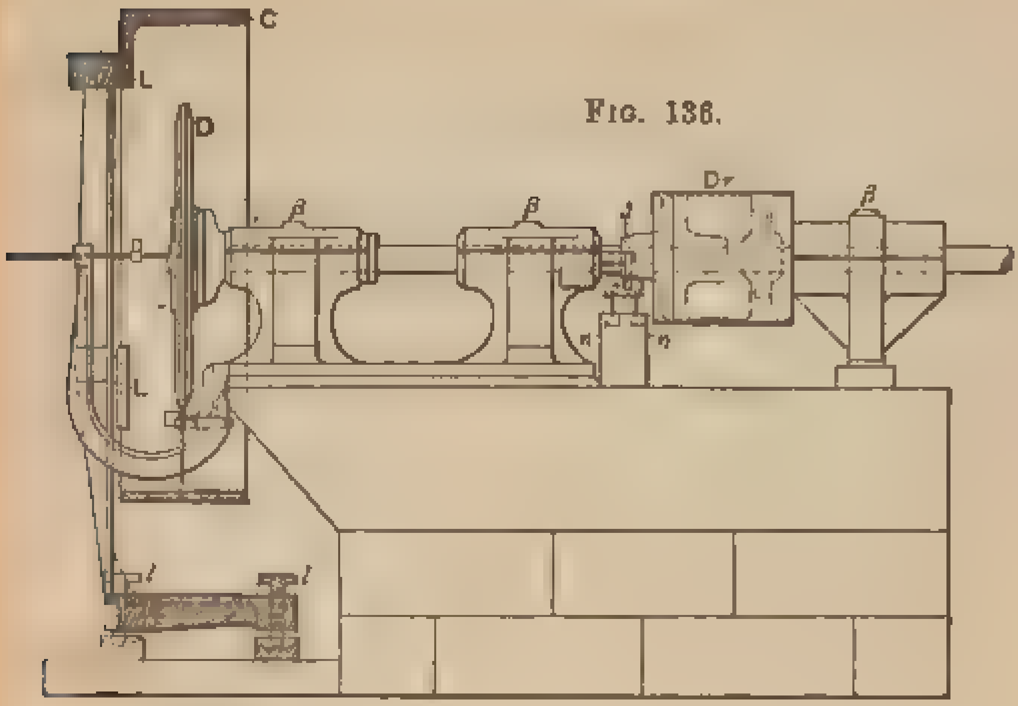
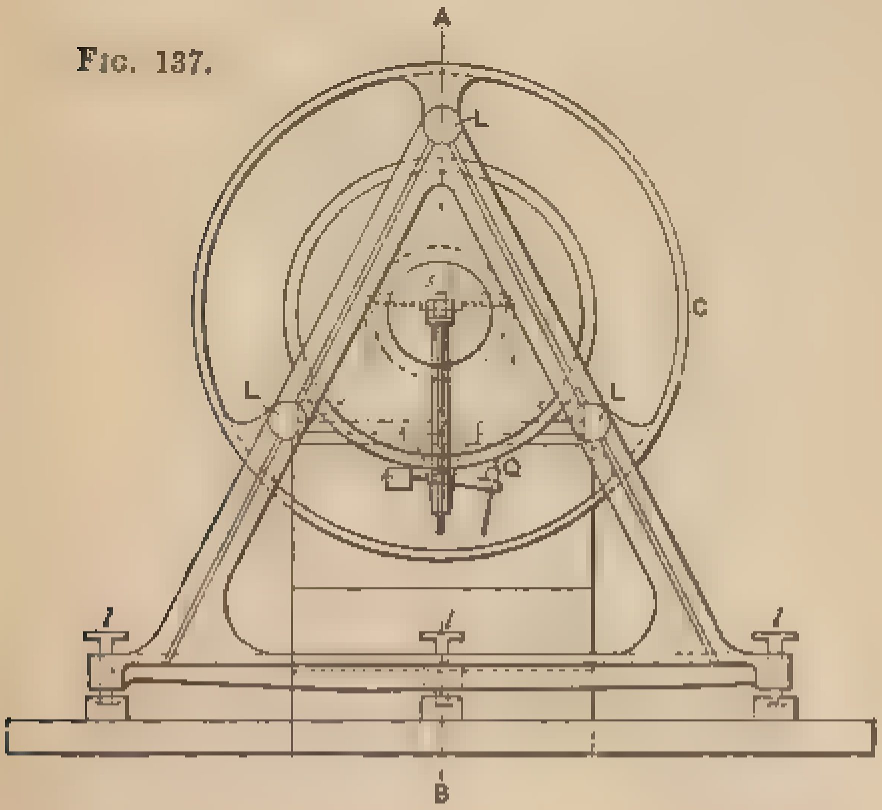


Fig. 137.



explanation of reference letters see Fig 135), is given in Fig. 136, and an end view showing the disk and edge-brush, Q, in Fig. 137.

The brush finally adopted for the edge of the disk was a single wire perforated by a channel, through which was supplied a small stream of mercury. A piece of copper an inch long was drilled to a depth of $\frac{3}{4}$ inch, to meet another hole at right angles, which received the phosphor wire brush. The perforation drilled along the wire of the brush was connected with that in the copper piece, and an india-rubber tube slipped over the free end of the latter kept up a constant supply of mercury. This gave a constantly fresh surface for contact. The central brush was fed also with mercury but more slowly.

Method of Measuring Speed of Disk.

The speed of driving was measured by the stroboscopic method by observing one of a set of rows of teeth, marked round a cylinder, through slits in brass plates attached to the prongs of a tuning fork, which vibrated at right angles to the circles of teeth The fork was bowed, not electrically maintained: the number of turns per second n was given as in Lord Rayleigh's experiments by $n = 2f/N$, where f is the frequency, and N the number of teeth in the stationary circle.

The pitch of the fork was determined by driving the cylinder; keeping a row of teeth stationary, and causing the cylinder by means of a lever to make and break a battery circuit every revolution, so that for about half the time of revolution the contact was made and for the other half broken This registered on a telegraph tape a series of alternate dashes and spaces, and on the same tape a mark was made once a second by the laboratory standard clock. The observations being continued over three or four minutes, N and n were obtained with accuracy, and f was deduced by the equation $f = \frac{1}{2}nN$.

Arrangement of Mercury Column.

The resistance used for balancing the electromotive force of the disk was a column of mercury, so that the experiment gave the specific resistance of mercury directly. The mercury was placed in a long rectangular trough, Fig. 138, carefully cut, as described below, in paraffin by machinery, and two electrodes dipped into the mercury at some distance from the ends of the trough One of these electrodes was kept fixed, the other was attached to the movable headstock of the Whitworth measuring-machine, by which its position was altered by the difference of distance between the electrodes necessary for two different speeds of the disk Thus the difference only of two distances between the electrodes (and this could be obtained with accuracy) was used in deducing the final result. For if n_1, n_2, be the two speeds of rotation of the disk, ρ the specific resistance of

mercury, A the cross-section of the column, and l the distance between the two positions of the electrodes,

$$\frac{l}{A}\rho = M(n_1 - n_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Elimination of Uncertainty from Capillarity.

The capillary depression at the sides of the trough was allowed for by taking observations for two different depths of mercury in the trough. For if ΔA be the change in area produced by increasing the depth from h to h', n_1, n_2, l the speeds of rotation and difference of lengths of column in the first case, n'_1, n'_2, l' those in the second, then we have by (37), assuming that the groove is true and the temperature the same in both experiments,

$$A = \frac{l\rho}{M(n_1 - n_2)},$$

$$A + \Delta A = \frac{l'\rho}{M(n'_1 - n'_2)},$$

and therefore

$$\Delta A = \frac{\rho}{M}\left(\frac{l'}{n'_1 - n'_2} - \frac{l}{n_1 - n_2}\right),$$

or since $\Delta A = b(h' - h)$, where b is the breadth of the trough and h', h, the two depths,

$$\rho = \frac{Mb(h' - h)}{\dfrac{l'}{n'_1 - n'_2} - \dfrac{l}{n_1 - n_2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Preparation of Trough for Mercury.

The trough (shown in section in Fig. 138) was cut in paraffin wax melted in a longitudinal groove left in a strong casting of iron. The wax was melted in the groove and allowed to solidify on the surface, after which melted wax was poured through a hole in the crust to the interior in order to obtain a perfectly homogeneous mass. A channel was then cut and covered with a thin layer of paraffin to fill up air-holes, after which it was recut and scraped true.

A length of 10 inches of the trough was used in the experiments, and this was carefully calibrated by internal callipers of special construction.

Determination of Position of Mercury Surface.

The position of the surface of the mercury was determined by placing a spherometer in a fixed position over the trough and screwing down the movable point until contact was indicated by the completion of a battery circuit through the mercury and point. The division on the head of the micrometer corresponded to 1/5040 inch, and the size of the head allowed of an estimation of tenths of a division. Successive measurements did not differ by more than 1/20 800 of an inch when the point was kept clean by being carefully wiped with filter paper, and sparking was prevented as far as possible by including a large resistance in the circuit and breaking the circuit before removing the point from the mercury after a reading.

The temperature of the mercury in the trough was determined by two thermometers, one at each end of the trough. A third

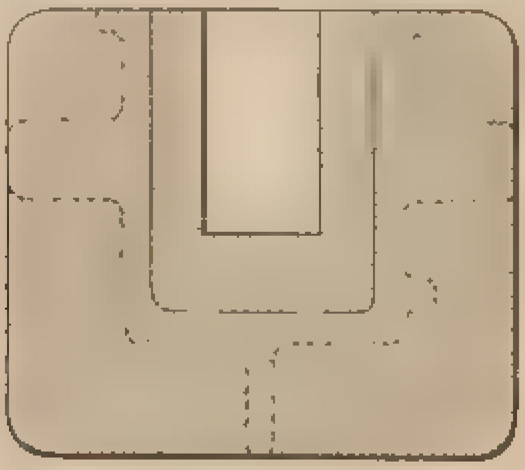


FIG. 138.

Observation of Temperature of Column, &c.

thermometer was placed between the prongs of the speed-measuring fork. These thermometers were corrected by comparison at Kew.

To prevent warping of the trough by change of temperature, and to make as certain as possible that the mercury in contact with the poorly conducting wax should be all at one temperature, the temperature was kept as nearly constant as possible by enclosing the trough, &c. in a wooden box covered with felt paper, and protected round about with felt curtains. The thermometers were read through windows in the box by lifting the curtain.

The galvanometer used to test for balance was a Thomson reflecting galvanometer of ·968 ohm resistance, the needle of which was carried by a quartz fibre 13 inches long.

The axis of rotation was placed at right angles to the magnetic meridian, so that the plane of the disk might be in the meridian and thus avoid any current due to earth induction. When the disk was rotated without current in the standard coil any displacement of the light spot could be annulled by a slight movement of a compensating magnet on the table.

Adjustment of the Disk in Position.

The bearings of the disk were made as nearly as possible perfectly true, and were each provided with a sight-feed lubricator. The disk was adjusted in position in the coil by arranging an arm to fit upon the disk so that a carefully scraped face on the arm should be a prolongation of the mean plane of the disk. The coil was then placed in position so that the outside edge of this face should travel round the interior circle cut in the mean plane of the coil as already described

Adjustment of Trough.

The mercury trough was carefully levelled and adjusted parallel to the bed of the measuring machine. The last adjustment was made by attaching to the movable headstock a cylinder projecting vertically downwards into the trough, running the headstock from end to end and testing at the extremities the distance from the cylinder to the same side of the groove by pushing a wooden wedge lightly between them. Further, by making the wedge-reading the same on both sides of the cylinder, the headstock was adjusted so that when an electrode was substituted for the cylinder it dipped into the medial plane of the mercury column.

A slight direct effect on the needle produced by the current was observed, and was compensated by placing a coil of three turns of the battery wire close to the needle.

The insulation of the wire of the coil from the bobbin and of the disk from the axle were tested and found satisfactory.

Mode of making Observations.

Lord Rayleigh's plan (p. 583 above) of taking two sets of galvanometer readings for each equilibrium position was followed. One set gave the change of galvanometer reading for reversal of current when the resistance was slightly below that required for balance, the other set the corresponding change when the resistance was a little above the proper current. To eliminate uncertainties owing to variations of speed and of the brush contacts, a number of reversals were quickly taken for each resistance and combined to give a mean result. The readings were taken without waiting for the needle to come to rest, but elongations were observed which with a previously determined damping coefficient enabled the position of rest to be calculated.

Calculation of M.

The dimensions of the coil and disk, and the calculation from them of the mutual inductance, M, are given at p. 314 above. It

only remains to state the mode of reduction of the observations and the final result.

Reduction of Results. If ρ_t and A_t be the specific resistance of mercury and the cross-section of the column at temperature t, and ρ, A, the same quantities at 15°·5, to which the results were in the first instance reduced, L the distance between the electrodes in any equilibrium position, then

$$Mn = \frac{L\rho_t}{A_t}.$$

Temperature Corrections. Now if f be the frequency of the fork at the standard temperature 15°·5, and f_θ the frequency at temperature θ, we have

$$n = \frac{2f_\theta}{N} = \frac{2f\{1 + k(\theta - 15^\circ{\cdot}5)\}}{N},$$

where k $(= -{\cdot}00011)$ was a temperature coefficient. Also

$$\rho_t = \rho\{1 + \alpha(t - 15^\circ{\cdot}5)\}$$
$$A_t = A\{1 + \gamma(t - 15^\circ{\cdot}5)\},$$

where α is the temperature coefficient for the specific resistance, and γ the coefficient cubical dilatation of mercury. Hence

$$Mn = \frac{2Mf\{1 + k(\theta - 15^\circ{\cdot}5)\}}{N} = \frac{L\rho\{1 + \alpha(t - 15^\circ{\cdot}5)\}}{A\{1 + \gamma(t - 15^\circ{\cdot}5)\}}$$
$$= \frac{L\rho}{A}\{1 + (\alpha - \gamma)(t - 15^\circ{\cdot}5)\},$$

or

$$\frac{L\rho}{A} = 2Mf\nu,$$

where

$$\nu = \frac{1 + k(\theta - 15^\circ{\cdot}5) + (\gamma - \alpha)(t - 15^\circ{\cdot}5)}{N}.$$

If now $\Delta\nu$ be the difference of two values of ν for equilibrium positions separated by an interval l,

$$\rho = 2MfAs$$

if $s = \Delta\nu / l$.

Two observations were made with the mercury at different levels h' and h to eliminate error from capillarity. Calling the two values of $\Delta\nu/l$ for these observations s', s, and the areas of cross-section of the trough A', A, we have, if b be the mean breadth of the trough over the length used,

$$A' - A = b(h' - h),$$

so that

$$\rho = \frac{2Mfb(k' - k)}{l s' - l_1 s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and from this the specific resistance of mercury at $15^\circ{\cdot}5$ was calculated.

The coefficient a was obtained from the formula

$$R_t = R_0(1 + \cdot 0008649t + \cdot 00000112t^2) \quad . \quad . \quad . \quad (40)$$

given by Mascart, de Nerville, and Benoit for the resistance of a column of mercury at t° in a glass tube. Thus

$$R_{15\cdot 5} = R_0 \times 1{\cdot}013675$$

and

$$R_t = R_0\{1 + (a - \beta)t\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

where β is the coefficient of cubical expansion of glass ($= \cdot 000008$). Thus

$$(a - \beta) \times 15{\cdot}5 = \cdot 013675.$$

or

$$a \times 15{\cdot}5 = \cdot 013799.$$

This gave the mean value of a from 0 to $15^\circ{\cdot}5$ which was used to obtain the specific resistance of mercury at 0° from its value at $15^\circ{\cdot}5$. The equation of reduction was thus

$$\rho_{15\cdot 5} = \rho_0 \times 1{\cdot}013 \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

The value of a at $15^\circ{\cdot}5$ or $a_{15\cdot 5}$ was obtained by calculating from (40) above

$$\frac{dR_t}{dt} = R_0 \times (\cdot 0008649 + \cdot 00000224 \times 15{\cdot}5)$$

$$= R_0 \times \cdot 00089962.$$

But by (41)

$$\frac{dR_t}{dt} = R_0(a_{15\cdot 5} - \beta)$$

and therefore

$$a_{15\cdot 5} = \cdot 0009076,$$

which was used to correct the experimental results for the small differences between $15^\circ{\cdot}5$ and the observed temperatures.

Final Result of Experiments.

The final result of five sets of experiments gave

$$\rho = 94067 \text{ C.G.S.}$$

as the resistance at 0° C. of a column of mercury one square centimetre in cross-section and one centimetre in length.

Realized Value of Ohm as Mercury Column

According to this result the ohm is equal to the resistance at 0° of a column of mercury 106·307 centimetres long and one square millimetre in cross-section.

Method of Joule.

Joule's method is in principle very simple. Supposing a current of strength γ to flow through a wire of resistance R for a time t, a quantity of energy $\gamma R^2 t$ is spent in the conductor. This is expressed in ergs if γ and R are taken in C.G.S. units and t in mean solar seconds of time. If H be the heat generated in the conductor in that time, then if J be the work equivalent of the unit of heat, we have

$$\gamma^2 Rt = JH$$

and

$$R = \frac{JH}{\gamma^2 t}.$$

The absolute measurement of the current might be made with sufficient accuracy, though it is of very nearly the same order of difficulty as the determination of the ohm; but there are also involved exact calorimetric determinations which require great care and skill. Over and above all these is the determination of J with an accuracy equal or superior to that to which it is required to find the ohm, say to 1 in 10000. This would be a research of difficulty far transcending that of the measurement of absolute resistance by most other methods.

For descriptions of other methods, the reader may refer to Wiedemann's *Elektricität*, Band 4, 2te Abth.

It has been proposed by Prof. Carey Foster * to modify the method of revolving coil by rotating the coil on open circuit and applying to its terminals at

* B A. Report, 1881.

Carey Foster's Modification of Method of Revolving Coil.

the instant when the inductive electromotive force is a maximum a difference of potential equal and opposite to that then existing at the terminals of the coil. This will not be exactly the instant at which the coil passes through the meridian, as on account of the capacity of the conductors a certain retardation of phase will exist.

This applied difference of potential may be that existing between the terminals of a conductor in which a current γ is flowing. The current is measured by a tangent galvanometer of principal constant G, and therefore has for absolute value $H \tan \alpha / G$; so that the applied difference of potential is $RH \tan \alpha / G$. The induced difference of potential has the value $AH\omega$ only. Assuming H to be the same for the revolving coil and the galvanometer, we have therefore

$$R = GA\omega \cot \alpha = 2\pi^2 nn'\omega \frac{a^2}{a'} \cot \alpha,$$

if a be the mean radius of the revolving coil, a' that of the galvanometer, n, n', the numbers of turns in the coils.

Thus error of measurement of the mean radius a is of twice the importance of equal proportional error in a'.

The main advantage of this method lies in the elimination of self-induction, as the current is almost zero at each instant. In its practical use error from thermo-electric force at the rubbing surfaces, and from mutual induction between the wire circuit and secondary circuits in the ring currents would have to be guarded against.

The method does not seem to have been applied to a complete determination of absolute resistance.

Table of Collected Results.

The following table extracted mainly from a Report on the 'Absolute Resistance of Mercury' by R. T. Glazebrook (*Brit. Assn. Report*, 1891), contains the principal results obtained since 1881 :—

Date.	Observer.	Method.	Value of B A unit in Ohms.	Value of Ohm in Cms. of Mercury.
1882	Lord Rayleigh and Schuster	Revolving Coil	·98651	106·24
1883	Lord Rayleigh and Mrs. Sidgwick	Method of Lorenz	·98677	106·21
1884	G. Wiedemann	Earth Inductor		106·19
1884	Mascart, de Nerville, and Benoit	Induced Currents	·98611	106·33
1887	Rowland	Mean of Several Methods .	·98644	106·32
1887	Kohlrausch	Damping of Magnet	·98660	106·32
1882 and 1888	Glazebrook	Induced Currents	·98665	106·29
1890	Wuilleumier	Induced Currents	·98686	106·27
1890	Duncan and Wilkes	Method of Lorenz	·98634	106·34
		Mean	·98653	
1890	J. V. Jones	Method of Lorenz	—	106·307
1884	*H. F. Weber	Induced Currents	—	105·37
—	*H. F. Weber...	Rotating Coil ...	—	106·16
1884	*Roiti .. .	Induced Currents	—	105·89
1885	*Himstedt	—	—	105·98
1883	*Wild	Damping of a Magnet...	—	106·03
1889	*Dorn .	Damping of a Magnet	—	106·24
1885	*Lorenz .. .	Method of Lorenz	—	105·93

It has been decided (1892) by the British Association Committee on Electrical Standards to define the ohm for practical purposes as the resistance at 0° of a uniform column of mercury weighing 14·4521 grammes, in a tube 106·3 cms. long. This corresponds to cross-section 1 sq. mm., and density of mercury 13·5956.

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* The absolute measurements here referred to were compared with standards of German silver by Siemens or Strecker. The values in mercury units of these standards were certified by the makers.

CHAPTER XI

COMPARISON OF UNITS

THE experimental comparison of the ordinary electrostatic and electromagnetic units of an electrical quantity is of great importance in the electromagnetic theory of light, as it enables the velocity of propagation, according to that theory, of an electromagnetic disturbance to be determined numerically, and compared with the observed velocity of light. To make clear how the ratio of the two units of the same quantity is related to the velocity of propagation of electromagnetic waves, we shall use here one or two illustrations due to Clerk Maxwell, modifying however the mode of applying them in accordance with the more general theory of dimensions adopted in Chapter VIII. above.

Ratio of Units.

It has been shown (p. 118) that the electromagnetic force acting on an element ds of a conductor carrying a current γ in a magnetic field is $\mathbf{B}\gamma \sin\theta ds$, if $\mathbf{B}$ be the magnetic induction at the element, and θ the angle between the element and the direction of the magnetic induction.

Illustrations of Velocity of Propagation of Electromagnetic Action.

If the field be produced by a current γ' on a straight conductor parallel to ds at distance b from it, we get by integration of the expression $\gamma' \sin\theta' ds'/r^2$ (p. 143 above) the expression $2\gamma'/b$ for the field intensity at ds

First Illustration.

due to the current γ', if the conductor in which it flows be infinitely long. Hence, if μ be the permeability of the medium, the electromagnetic force on ds is $2\mu\gamma\gamma' ds/b$; and if the first conductor be straight the force on a length $b/2$ is $\mu\gamma\gamma'$.

Now let the quantities of electricity γt, $\gamma' t$, conveyed by the currents in time t, be used to charge two spheres whose centres are at a distance r apart great in comparison with the radius of either. The electrostatic repulsion between the spheres would then be $\gamma\gamma' t^2/Kr^2$, if K denote the electric inductive capacity of the medium. If r be chosen so that this force is the same as the attraction between the conductors exerted on a length equal to half the distance between them, we have

$$\mu\gamma\gamma' = \frac{\gamma\gamma' t^2}{Kr^2},$$

or

$$\frac{1}{\mu K} = \frac{r^2}{t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

that is, $1/\sqrt{\mu K}$ may be expressed as a velocity. This is true whatever hypothesis as to dimensions is adopted for μ and K.

This velocity, moreover, is perfectly definite. For, if t^2/r^2 remain constant, the electrostatic force of repulsion between the spheres will remain unchanged, while their charges are increased at the time-rates γ, γ', respectively; and, therefore, $1/\sqrt{\mu K}$ is equal to the velocity with which the spheres must be separated in order that their mutual repulsion may then remain

equal to the force of attraction on a length of either of the parallel conductors equal to half the distance between them. It has been shown, p. 200 above, that $1/\sqrt{K\mu}$ is the velocity of propagation of an electromagnetic wave in an isotropic insulating medium.

Ratio of Units of Quantity considered as a Velocity.

If now we denote by v the ratio of the electromagnetic to the electrostatic unit of quantity, the charges on the spheres expressed in ordinary electrostatic units are, if γ, γ', now denote the ordinary electromagnetic measure of the currents, $v\gamma$, $v\gamma'$. Hence the force between the two spheres is

$$\frac{v^2\gamma\gamma' t^2}{K_s r^2},$$

where K_s denotes the specific inductive capacity of the medium, defined in the ordinary way as the ratio of the electric inductive capacity to that of the medium of reference (air or vacuum for example). But if μ_m denote the ordinary electromagnetic value of the permeability,

$$\mu_m \gamma\gamma' = v^2 \frac{\gamma\gamma' t^2}{K_s r^2},$$

that is

$$v^2 = \mu_m K_s \frac{r^2}{t^2},$$

or by (1),

$$v^2 = \frac{\mu_m K_s}{\mu K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the medium be air, for which $K_s = 1$ $\mu_m = 1$, we have

$$v = \frac{1}{\sqrt{\mu K}} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or v is equal to the velocity of propagation of an electromagnetic disturbance in air.

Second Illustration.

The following illustration, also due to Maxwell, gives a remarkable physical meaning to the velocity $1/\sqrt{\mu k}$ of propagation of an electromagnetic disturbance. In the first place it is assumed that an electrified surface in motion may be regarded as equivalent to a current.

This assumption is justified by the experiments of Rowland, who has found that a statically electrified surface set into rapid motion affects a magnet properly placed in its vicinity, and has made measurements of the magnitude of the effect produced.

Repulsion between Two Electrified Surfaces.

Considering then a plane surface of indefinite extent electrified to a surface density σ taken in any chosen system of units, we have $u\sigma$ as the measure of the convection current across unit breadth at right angles to the direction of motion, if u be the velocity. Let now another surface parallel to the first and at a distance b from it be electrified to a uniform density σ', and move with velocity u', in the same direction as in the former case. A current in this case of strength $u'\sigma'$, per unit of breadth of the electrified surface, may be regarded as flowing parallel to the former current.

The two surfaces will repel one another electrostatically and attract one another electromagnetically. The electrostatic repulsion between two elements of surface dS, dS', at distance r is $\sigma dS.\sigma' dS'/Kr^2$, and integrating over the first surface we get $2\pi\sigma\sigma' dS'/K$ for the resultant force on an element dS' of the second surface. Hence the force over unit area is $2\pi\sigma\sigma'/K$.

Attraction between Two Moving Electrified Surfaces, regarded as Two Plane Current Sheets.

The electromagnetic force between the two plane current sheets can be found as follows. Consider two narrow strips of the two planes in the direction of motion. Let dz, dz', be their breadths, and z' the distance of the second strip from a plane coinciding with the first strip, and cutting the two moving plane surfaces at right angles. The distance between the two strips is $\sqrt{b^2 + z'^2}$. The attraction between them is $\mu u\sigma dz\, 2u'\sigma' dz'/\sqrt{b^2 + z'^2}$ per unit of length of either. The total attraction, F say, per unit of length on the strip of breadth dz, is at right angles to the planes, and can be found by resolving the attraction just found in that direction, and integrating from $z' = -\infty$ to $z' = +\infty$. Thus

$$F = 2\mu uu'\sigma\sigma' b\,dz \int_{-\infty}^{+\infty} \frac{dz'}{b^2 + z'^2}$$
$$= 2\pi\mu uu'\sigma\sigma' dz.$$

Thus the electromagnetic attraction on unit area of either plane is $2\pi\mu u u'\sigma\sigma'$.

If the electrostatic repulsion be supposed to balance the electromagnetic attraction and u be taken equal to u', we get

$$\frac{2\pi\sigma\sigma'}{K} = 2\pi\mu u^2\sigma\sigma'$$

or

$$u^2 = \frac{1}{\mu K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus the velocity of propagation of an electromagnetic disturbance in the medium is equal to the velocity with which the two electrified planes must move relatively to the medium in order that there may be no mutual force between them.

It has been shown above (p. 535) that v may be obtained from the ratio of the electrostatic and electromagnetic measures of any electric or magnetic quantity. It has been found experimentally in at least six of the following different ways:— Methods of Determining v.

I. By measuring electrostatically and electromagnetically a given quantity of electricity.

II. By measuring electrostatically and electromagnetically a given difference of potential.

III. By comparing the value of the electrostatic capacity of a given standard condenser, obtained by calculation from its dimensions and arrangement, with its capacity in electromagnetic measure as given by experiment.

IV. By comparing an electrostatic capacity, obtained by calculation as in III., with the self-inductance of a coil.

V. By determining (in either system of units) the product CL of the capacity of a given condenser, and

the self-inductance of a given coil, and comparing this with the product of the electrostatic value C_s of the capacity and the electromagnetic value L_m of the self-inductance. [The product CL is the same in both systems of units.]

VI. By measuring electrostatically and electromagnetically a given resistance.

VII. By observation of the period of oscillatory discharge of a condenser of known capacity (in electrostatic units), through a circuit of known self-inductance.

Experiments of Weber and Kohlrausch.

The first attempt to determine v was made by Weber and Kohlrausch, who employed method I.* A Leyden jar was charged to a potential measured electrostatically by means of an electrometer, and was then discharged through a ballistic galvanometer, which measured by the throw of the needle the quantity of electricity with which the jar was charged. This quantity was known in electrostatic measure from the measured potential and the capacity of the jar, which was obtained by comparison with that of a sphere insulated at a distance from other conductors. The value obtained for v was 31,074,000,000 cms. per second.

This determination cannot be regarded as one of high accuracy, chiefly on account of the unsuitableness of a condenser with a solid dielectric for exact experiment. The construction also of absolute electrometers for exact work had not then been brought to so high a pitch of excellence as has since been reached.

A determination by this method with the most

* *Abh. d. Königl. Sächs. Ges. d. Wissens.* 1856.

refined appliances has since been carried out by Professor Rowland at Baltimore,* and of this we give here a more detailed account.

Rowland's Experiments.

The electrometer employed was an absolute instrument made on Thomson's guard-ring principle. The protected disk was 10.18 cms. in diameter, and was suspended in an aperture in the guard-ring of 1 mm. greater radius.

Details of Electrometer.

The diameters of the guard-plate and attracting-plate were each 330 cms. The surfaces were all nickel plated, and worked true, so that the distance between the surfaces could be accurately found. The disk could be adjusted in the plane of the guard-ring, and the attracting-plate and disk to parallelism, to $\frac{1}{10}$ mm. External action was screened from the disks by a case of sheet brass.

The protected disk was hung from one arm of a sensitive balance, and the exact position of the beam was observed by means of a hair moving in front of a scale in the manner described above (Vol. I. p. 263).

In the actual use of the electrometer, since the suspended disk could not be in stable equilibrium under the action of electrostatic attraction, its swing was limited to a range of $\frac{1}{10}$ mm. on each side of the sighted position; and the attracting plate then placed at two near positions, for one of which the plate rose above the sighted position, for the other fell below it. The mean of these was taken as the reading for the position of the attracting-plate.

Mode of Using Electrometer.

If d be the distance of the electrometer plates apart, w the weight on the balance, and S the area of the disk, we have (see Vol. I. p. 58) for the electrostatic measure, V_s, of the difference of potential between them

$$V_s^2 = \frac{8\pi d^2 wg}{S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and by a formula given by Maxwell † for the effective area of a protected disk of radius R, in an opening of radius R',

$$S = \tfrac{1}{2}\pi\left\{R'^2 + R^2 - (R'^2 - R^2)\frac{a}{d + a}\right\} \quad . \quad . \quad . \quad (6)$$

where $a = (R' - R)(\log 2)/\pi = \cdot 221\,(R' - R)$ nearly.

* *Phil. Mag.* Oct. 1889.

† *El. and Mag.* vol. i. Art. 201.

Thus the working equation for V_s was

$$V_s = 17221\, d\sqrt{\kappa}\left(1 + \frac{\cdot 0002}{d}\right). \quad . \quad . \quad . \quad . \quad (7)$$

Standard Condenser.

The standard condenser consisted of two concentric spheres. The spheres were very accurately constructed, and the inner was hung concentrically within the outer by a silk cord (see also p. 629 below). Two balls of different diameters were provided for use as inner spheres. The electrostatic capacity was obtained by determining the diameters of the balls by weighing in water, and was 50·069 C.G.S. or 29·556 C.G.S. according as the larger or smaller inner sphere was used.

Galvanometer.

The galvanometer used for the discharges was a specially constructed and carefully insulated instrument. It had two coils, each of about 5600 turns of No. 36 silk-covered copper wire. These were fixed on the two sides of a plate of vulcanite. The needle was surrounded by a metal box to screen off possible electrostatic action of the coils from the needle.

Determination of Constant of Galvanometer.

The constant of this galvanometer was determined by comparison with the galvanometer described above (p. 547). The constant of this had been slightly altered, and was now found to be by measurement of its coils 1832·24, by comparison with an electrodynamometer 1833·67, and by comparison with a single circle (p. 548) 1832·56, giving a mean of 1832·82 instead of 1833·19 as before. The ratio of the constant of the new galvanometer to this was found to be 10·4141, so that for the ballistic galvanometer used

$$G = 19087,$$

including the factor for the number of turns.

Determination of H at Ballistic Galvanometer.

An absolute electrodynamometer on Helmholtz's double-coil principle, similar to that described at p. 365 above, was used to find the directive force H at the ballistic galvanometer, at any instant during the progress of the experiment, so as to eliminate magnetic changes which were continually going on in the building used for the investigation; changes which were all the more important as H was only about $\frac{1}{3}$ of the horizontal component of the earth's field at the place. The suspension of the instrument was a bifilar one, and it was found that no correction was necessary for the torsion of the wire.

It follows from (10), p. 368 above, that if c be a constant depending on the coils, and the electrodynamometer be set up so

that H does not affect it, or readings be taken so as to eliminate it, and the same current pass through both coils, we may write

$$\gamma = c\sqrt{F}\sqrt{\sin\beta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where F is the coefficient of $\sin\beta$, in the couple applied by the bifilar, β being the angle through which the suspension head is turned to bring the suspended coil back to parallelism with the fixed coil. But it is clear that, if mk^2 be the moment of inertia of the coil, by the theory of simple harmonic motion we have $\ddot{\beta}/\beta = -4\pi^2/T^2$, and $\ddot{\beta} = -F\sin\beta/mk^2$, so that

by Employment of Electrodynamometer.

$$\sqrt{F} = \frac{2\pi}{T}\sqrt{mk^2}.$$

Thus, including 2π in the constant c, we have

$$\gamma = \frac{c\sqrt{mk^2}}{T}\sqrt{\sin\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

for the electrodynamometer.

The value of c was calculated from the particulars of the coils which were

Constant of Electrodynamometer.

	Large Coils.	Suspended Coils.
Mean radius	13·741 cms.	2·760 cms.
Mean distance	13·786 „	2·707 „
Radial depth	·84 „	·41 „
Axial width	·86 „	·38 „
No. of turns	240	126

From which by (10) p. 368 above, and the values of G_1, g_1, given at p. 269

$$c = \cdot 012914.*$$

To verify this constant a circle 80 cms. in diameter was made and used as the coil of a tangent galvanometer. The ballistic galvanometer was set up so that its needle was at the centre of this circle, and acted, when required, as the suspended needle of the tangent galvanometer of which the circle was the coil. The current from the electrodynamometer was passed through the circle, and the horizontal field intensity H deduced from the

* This is double the value given by Prof. Rowland in his paper. The full period of vibration appears in equation (7), whereas Prof. Rowland used the half period.

galvanometer deflection and the current as given by the electrodynamometer. The value of H was found also by the magnetic method, and the two results were found to differ by only about 1 in 1000. Thus the tangent galvanometer gave $\frac{1}{2}c = \cdot006451$, and the mean $\cdot006454$ of this and the former result was used.

The moment of inertia mk^2 was found by placing weights at different distances along a tube passed through the centre of the suspended coil, and observing the period of free swing of the coil. It was thus found that $mk^2 = 826\cdot6$ in gramme-centimetre units.

Method of Determining H. The value of H at the needle of the ballistic galvanometer was found, when required, by sending the same current through the dynamometer and the galvanometer, observing the deflections in the two cases, calculating the value of the current from the deflection from the former, and hence deducing H by the tangent galvanometer formula.

Method of Experimenting. The condenser was charged by being connected to a large charged battery of Leyden jars. This battery was kept connected to the electrometer. The potential reading was first observed, then the battery connected to the condenser for an instant, after which the condenser was disconnected from the Leyden jar battery and discharged through the ballistic galvanometer. This was repeated 1, 2, 3, 4, or 5 times in succession, so that the galvanometer received that number of very nearly equal impulses in the same direction before it had moved far from the position of rest. The reading of the position of the electrometer attracting disk was again taken after the series of impulses, on disconnection of the battery from the condenser, and was slightly less than before of course. Corrections for the displacements of the needle from zero at the times of the successive impulses were calculated and applied.

The mean of the electrometer readings before and after a single discharge was, with a correction, taken as the potential of that discharge. This correction arose from the fact that the first reading was higher than that for the potential of discharge by a certain small amount depending on the capacities of the battery of jars and the condenser. It was obtained by multiplying the mean reading d of distance between the plates by a factor $1 - \cdot0013$, when the larger sphere was used in the condenser, and by the factor $1 - \cdot0008$ when the smaller sphere was used. The other series were similarly corrected.

A correction was applied for the time occupied in producing the series of impulses. This was calculated approximately on the supposition that the time between one impulse and the next was $\frac{1}{4}$ of a second, and without taking into account the altered

position of the magnet relatively to the coil or the induced magnetism of the needle. The inclination, however, of the magnet to the plane of the coil would cause the impulsive couple on the needle to be less for impulses later than the first, while the induced magnetization of the needle brought about by the same inclination would have an opposite effect. Prof. Rowland came to the conclusion by experiment that no sensible error from neglect of these refinements of correction could result.

Correction for Time occupied by Impulses on Needle.

The principal equations used in reducing the results were (7) above, and others obtained as follows:—

Reduction of Results.

First, the ballistic galvanometer equation for the quantity, Q (in electromagnetic units), of electricity discharged, is

$$Q = \frac{HT}{\pi G}(1 + \tfrac{1}{2}\lambda)\sin\tfrac{1}{2}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where θ is the ballistic deflection, corrected for everything except damping.

But if C_s be the capacity of the condenser in electrostatic units, and N the number of discharges,

$$Q = N\frac{V_s C_s}{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Also H was obtained from the constant current measured by the dynamometer while it flowed round the 80 cms. circle, at the centre of which the ballistic galvanometer needle was situated. Thus denoting by ϕ the deflection of the needle produced by the constant current, by r the radius of the large circle, and by b the distance of its plane from the centre of the ballistic needle, we have by (9) and the elementary theory of the tangent galvanometer

$$\gamma = \frac{c\sqrt{mk^2}}{T}\sqrt{\sin\beta} = \frac{(r^2 + b^2)^{\frac{3}{2}}}{2\pi r^2} H \tan\phi,$$

so that

$$H = \frac{2\pi r^2 c\sqrt{mk^2}\sin\beta}{T(r^2 + b^2)^{\frac{3}{2}}\tan\phi} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Using this in (10), equating to (11), and solving for v we find

$$v = \frac{(r^2 + b^2)^{\frac{3}{2}} G N V_s C_s \tan\phi}{r^2 c\sqrt{mk^2}\sin\beta\,(1 + \frac{1}{2}\lambda)\sin\frac{1}{2}\theta} \quad . \quad . \quad . \quad . \quad (13)$$

where V_s is given by (7), and C_s by the dimensions of the condenser.

Deduction of Angular Deflections from Readings.

The approximate equation

$$2 \sin \tfrac{1}{2}\theta = \tfrac{1}{2} \frac{\delta}{D}\left(1 - \tfrac{1}{3} \frac{\delta^2}{D^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

was used to find the value of $\sin \frac{1}{2}\theta$ from the observed deflection δ and the scale distance D. This approximation is easily obtained as follows: since

$$\delta/D = \tan 2\theta = 2 \sin \theta \cos \theta/(1 - 2 \sin^2\theta),$$

or

$$\sin \theta = \tfrac{1}{2} \frac{\delta}{D} \frac{1 - 2 \sin^2\theta}{\sqrt{1 - \sin^2\theta}}.$$

Putting $\sin \theta = \frac{1}{2} \delta/D$ on the right the equation becomes approximately

$$\sin \theta = \tfrac{1}{2} \frac{\delta}{D} \left(1 - \tfrac{3}{8} \frac{\delta^2}{D^2}\right)$$

or

$$2 \sin \tfrac{1}{2} \theta = \tfrac{1}{2} \frac{\delta}{D}\left(1 - \tfrac{3}{8} \frac{\delta^2}{D^2}\right) \frac{1}{\sqrt{1 - \sin^2 \frac{1}{2}\theta}}.$$

In the last factor on the right which is not very different from unity $\sin \frac{1}{2}\theta$ may be put equal to $\delta/4D$. The equation then becomes

$$2 \sin \tfrac{1}{2} \theta = \tfrac{1}{2} \frac{\delta}{D}\left(1 - \tfrac{3}{8} \frac{\delta^2}{D^2}\right) \left(1 + \tfrac{1}{32} \frac{\delta^2}{D^2}\right) = \tfrac{1}{2} \frac{\delta}{D}\left(1 - \tfrac{11}{32} \frac{\delta^2}{D^2}\right)$$

$$= \tfrac{1}{2} \frac{\delta}{D}\left(1 - \tfrac{1}{3} \frac{\delta^2}{D^2}\right), \text{ nearly.}$$

The value of $\tan \phi$ was calculated, by successive approximation from the value of $\tan 2\phi$ given by δ_1 and the distance D_1 of the scale from the mirror, so that

$$\tan \phi = \tfrac{1}{2} \frac{\delta_1}{D_1} \left(1 - \tfrac{1}{4} \frac{\delta_1^{\,2}}{D_1^{\,2}} + \tfrac{1}{8} \frac{\delta_1^{\,4}}{D_1^{\,4}}\right) \quad . \quad . \quad . \quad (15)$$

The following are the results obtained :—

Number of Discharges.	Mean result in cms. per second	Number of results of which mean was taken.
1	$298{\cdot}80 \times 10^8$	9
2	$298{\cdot}48 \times 10^8$	5
3	$297{\cdot}26 \times 10^8$	5
4	$297{\cdot}15 \times 10^8$	5
5	$296{\cdot}69 \times 10^8$	5

Final Results.

To these were given weights inversely as the number of discharges, except in the case of the first which was given twice the weight of the second, on account of the larger number of observations. Thus the final result obtained was $v = 2{\cdot}9815 \times 10^{10}$ in cms. per second.

Determinations by method II. which is due to Lord Kelvin (Sir W. Thomson) have been made by Lord Kelvin himself,* Mr. D. McKichan,† F. Exner,‡ and Mr. R. Shida.§

Lord Kelvin's Method.

A current is made to flow through a coil the absolute value R of the resistance of which is known, and the current is measured electromagnetically by an absolute current-meter, while the difference of potential between the extremities of the coil is measured by an absolute electrometer. If V be the difference of potential in electrostatic measure, the work done in the passage of one electrostatic unit of electricity is V. But one electrostatic unit of electricity is $1/v$ of an electro-

* *Phil. Trans. R S.* 1868.
† *Ibid.* 1879.
‡ *Wien. Ber.* 86, 1882.
§ *Phil. Mag.* 10, 1880.

magnetic unit; and if γ be the measured current, the time t taken for a quantity $1/v$ of electricity to pass is $1/v\gamma$. Hence the work done in the conductor or $\gamma^2 Rt$ is $\gamma R/v$. Thus

$$v = \frac{\gamma R}{V} \quad \ldots\ldots\ldots \quad (16)$$

The result therefore involves the absolute value of a resistance R in electromagnetic units. Now in the earlier experiments by this method the resistance of a conductor was not known with accuracy, and the results are unreliable, unless some means exists of converting the values of R which were used.

Results. Lord Kelvin's first result (corrected for the value of the B.A. unit) was $2{\cdot}808 \times 10^{10}$ cms. per second, Mr D. McKichan's $2{\cdot}896 \times 10^{10}$ cms. per second.

Shida's Experiments. Shida's determination was made later and gave $v = 2{\cdot}955 \times 10^{10}$ cms. per second. The difference of potential at the terminals of a battery of large tray Daniell cells was measured by a Thomson's absolute electrometer, while the current maintained by the battery through a tangent galvanometer was measured.

In reducing his results Mr. Shida multiplied both numerator and denominator of (16) by the factor $(R + r)/R$, where r was the resistance of the battery and connections. On this account the accuracy of the result was mistakenly called in question. For though the factor $R + r$ was of uncertain value, its introduction in both numerator and denominator could in no way affect the value of the ratio $\gamma R/V$. The real ground for uncertainty lay in the construction of the tangent galvanometer, which could barely work up to the degree of accuracy required.

A measurement of v was made by this method again in 1889 by Lord Kelvin, who used an improved absolute electrometer of

his own invention; but the details of the investigation do not seem yet to have been published. The result obtained was Lord Kelvin's later Experiments.

$$v = 3\cdot004 \times 10^{10} \text{ cms. per second.}$$

Exner's result obtained by a modification of this method was with the value ·941 ohm for one Siemens' unit, $2\cdot92 \times 10^{9}$ cms. per second.

Another form of this method has been given by Maxwell,* and used by him in a determination of v. The electromagnetic repulsion between two parallel coils produced by the same current flowing in opposite directions through them, was balanced by the attraction between two disks to the backs of which the coils were attached, and between which a difference of potential was produced by another current the ratio of which to the former current was known. One of the disks was the protected disk of a Thomson's guard-ring condenser, and to the back of this one of the coils was attached directly: the other coil was carefully insulated from the attracting disk by a plate of glass and a layer of insulating material. Maxwell's Method.

The apparatus is shown in Fig. 139, and shortly described in the list of references attached. The small disk (diameter four inches) and attached coil were carried at one end of a torsion balance suspended by a No. 20 copper wire from a graduated torsion head movable by a tangent screw. The disk and coil were protected by a cylindrical brass box 7 inches in diameter, one end of which formed the guard-ring. The disk carried on the side towards the interior of the box a glass scale divided to $\frac{1}{100}$ of an inch, which was viewed by a reading microscope fixed on the outside of the box.

To eliminate the turning couple due to the earth's field a coil was attached to the other end of the balance, and connected with the first coil in such a way that the current flowed through the coils in opposite directions.

* *Phil. Trans. R.S.* 158 (1868), or *Rep. of Papers*, Vol. II. p. 125.

The attracting disk (which was 6 inches in diameter) was, with its attached coil, on a slide worked by a micrometer so that the distance of the disks could be varied and measured. The plane of this disk was adjusted parallel to the guard-ring, which was placed exactly vertical by means of adjusting screws.

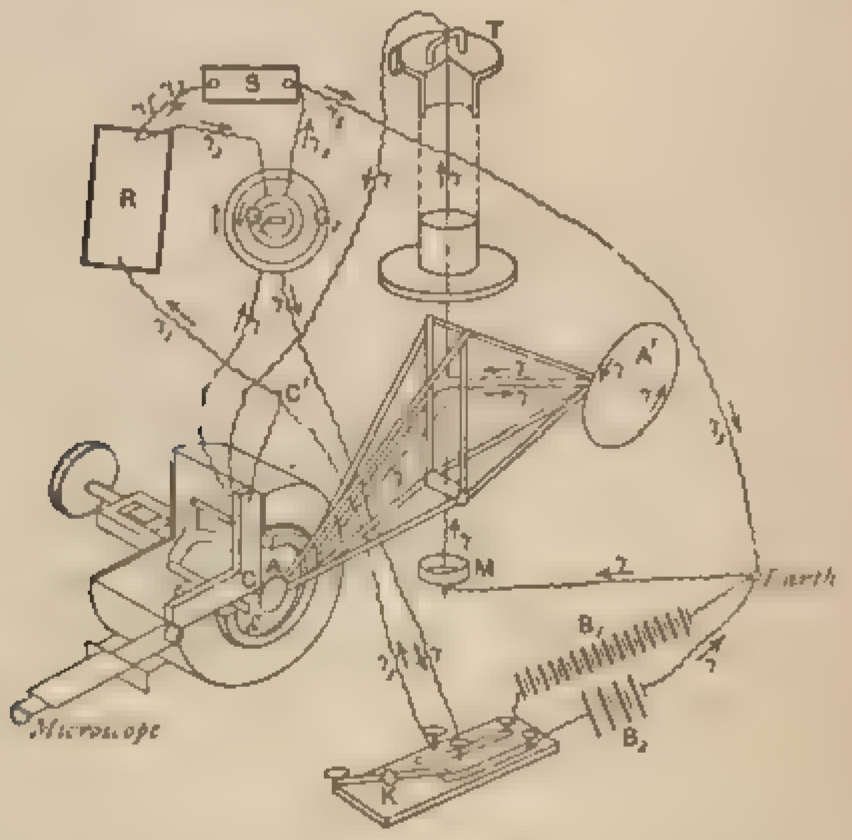

FIG. 139.

A	Suspended disk and coil	K	Double key
A'	Counterpoise disk and coil	G.	Graduated glass scale
C	Fixed disk and coil.	C'.	Electrode of fixed disk
B_1	Great battery	γ.	Current through the three coils and G_2
B_2.	Small battery	γ_1.	Current through R
G_1	First coil of Galvanometer	γ_2.	Current through G_1.
G_2	Second coil of Galvanometer.	$\gamma_1 - \gamma_2$.	Current through S
R	Great resistance		
T.	Torsion head and tangent screw.		

Adjustment of Apparatus. The graduations of the glass scale and the micrometer were compared by pressing the suspended disk forward by a light spring against the large disk, and then working the screw so as to send the small disk back towards the plane of the guard-ring, while readings of the micrometer were taken for successive

divisions of the glass scale. This motion was quite regular until the large disk came into contact with the guard-ring at one point. It was found then that a motion of about $\frac{1}{1000}$ of an inch sufficed to bring the whole of the guard-ring into contact with the large disk.

When the small disk had thus been brought into the plane of the guard-ring, the reading microscope had its cross-wires focussed on a known division of the glass-scale, and two pieces of silvered glass were fixed, one to the back of the guard-ring, the other to the back of the suspended disk, so that when the disk and guard-ring were in one plane these mirrors were also, and gave a continuous image of objects in front of them. This arrangement gave a test of coplanarity of the surfaces to $\frac{1}{1000}$ of an inch. Adjustment of Guard-ring and Disk to Coplanarity.

The torsion wire, which was of soft copper stretched to straightness, seemed in great measure free from imperfectness of elasticity. The torsion balance could be adjusted by moving the supporting pillar, which could be adjusted and clamped in position by screws at its base. The balance itself could be raised or lowered, turned about any horizontal axis by sliding weights attached to it, and about the axis of suspension by the torsion head.

A large battery, the property of Mr. Gassiot, containing 2600 cells charged with bichloride of mercury, was used to electrify the disks. One terminal of the battery was connected through a key with the large disk, the other with the case of the instrument, and the circuit between was composed of a large resistance of over a megohm, in series with one (hereafter called the first) coil of a standard galvanometer shunted by a coil of resistance S. Arrangement of Currents.

A current was sent from another battery through a second coil of the tangent galvanometer (in the direction opposed to the other coil), through the coil behind the large disk, and thence to the suspended coils by the suspension wires. A common connection was given to earth, the case, and the other electrode of the battery, by a copper wire hanging from the centre of the torsion balance, and dipping into a mercury cup.

When the suspended disk was at rest at zero the battery contacts were made simultaneously, and, according as the suspended disk was attracted or repelled, the other was moved farther from or nearer to the suspended one. It was necessary, on account of the instability of the small disk, when at the zero position under the action of the electric forces, to work the micrometer disk gradually up by successive trials from a distance initially too great, making contacts as zero was approached, so as if possible Mode of Experimenting.

to bring the suspended disk to rest under the action of the opposing forces due to the disks and coils. An observer at the galvanometer altered the shunt S, while the contacts were being made, so as to bring the needle to zero.

Comparison of Coils of Galvanometer. To compare the magnetic effects produced by the two galvanometer coils at the needle, a current was sent through the second coil of the galvanometer, then through a divided circuit, consisting of a resistance of 31 B.A. units placed across a branch made up of the first coil of the galvanometer and an added resistance S'. The latter resistance was varied until the effects on the needle balanced one another.

Theory of Method If V denote in electrostatic units the difference of potential between the disks, a the radius of the small one, and b their distance apart, the attraction between them was, clearly,

$$\frac{1}{2}\frac{V}{b}\frac{V}{4\pi b}\pi a^2 = \frac{1}{8}V^2\frac{a^2}{b^2}.$$

The repulsion between the two coils is $\gamma^2 dM/dx$, if γ be the current in each, x the distance apart of their mean planes, and M their mutual inductance. Thus we have

$$\frac{1}{8}V^2\frac{a^2}{b^2} = \gamma^2\frac{dM}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

But the difference of potential, V, between the disks is produced by the large battery, which sends a current γ_1 through the resistance R, and a current $\gamma_1 S/(G+S)$, ($=\gamma'$, say), through the first coil of the galvanometer, if G denote the resistance of that coil. Hence if E be the electromagnetic measure of this difference of potential

$$E = \left(R + \frac{GS}{G+S}\right)\gamma_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Again if F_1, F_2, be the magnetic forces produced at the needle by unit current in the two coils, we have $F_1\gamma' = F_2\gamma$, or $F_1\gamma_1 S/(G+S) = F_2\gamma$. But if in the comparison of the magnetic forces which was made γ'_1, γ'_2, denote the currents in the two coils, $F_1\gamma'_1 = F_2\gamma'_2$, and by the arrangement of the circuits $(G+S')\gamma'_1 = 31(\gamma'_2 - \gamma'_1)$, so that $F_2/F_1 = 31/(G+S'+31)$. This substituted in the former equation gives

$$\gamma_1 = \frac{G+S}{S}\,\frac{31}{G+S'+31}\,\gamma,$$

and (18) becomes, with this value of γ_b,

$$E = \left(\frac{RG}{S} + R + G\right)\frac{31}{G + S' + 31}\gamma. \quad . \quad . \quad (19)$$

But if γ_m, γ_s, denote the electromagnetic and electrostatic values of the same current, $E\gamma_m = V\gamma_s$ since they denote the same rate of working: and we have $v\gamma_m = \gamma_s$. Hence $V = E/v$. Substituting this value of V_m in (17) with that of E given by (19), and solving for v, we get

$$v = \frac{1}{2\sqrt{2}}\left(\frac{RG}{S} + R + G\right)\frac{31}{G + S' + 31}\frac{a}{b}\frac{1}{\sqrt{\frac{dM}{dx}}}. \quad (20)$$

The value of dM/dx given in terms of elliptic integrals at p. 402 above was used in the calculation of v by this formula. The numbers of turns in the coils were 144 and 121, and their mean radius was 1·934 inch.

The mean of 17 experiments gave Final Result.

$$v = 2{\cdot}8798 \times 10^{10}, \text{ in cms. per second,}$$

on the assumption that 1 B.A. unit was 10^9 C.G.S. The corrected result is

$$v = 2{\cdot}841 \times 10^{10}, \text{ in cms. per second,}$$

if 1 B.A. unit be taken as ·98674 ohm.

Method III. has been used by Professors Ayrton and Perry, J. J. Thomson, E. B. Rosa, and others. Third Method of Determining v

If C_m be the capacity of the condenser in electromagnetic units determined by any process, and C_s its capacity in electrostatic units as given by measurement, then if Q_m and Q_s denote the electromagnetic and electrostatic values of the same charge, we have $Q^2{}_m/C_m = Q^2{}_s/C_s$, since each denotes the same quantity of electric energy. Thus

$$\frac{C_m}{C_s} = \frac{Q^2{}_m}{Q_s^2} = \frac{1}{v^2},$$

or

$$v = \sqrt{\frac{C_s}{C_m}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Ayrton and Perry's Experiments.

The arrangement of Ayrton and Perry's apparatus * is shown in Fig. 140. The attracting plate P of a guard-ring condenser was connected to a key K, by which it would be put in contact with either terminal, A or B, of a resistance of about 10000 ohms. Unless the key was depressed it was kept in contact with B by means of a spring. The resistance was in circuit with a battery of 382 Daniell's cells, and the point B was connected with the earth and with the guard ring as shown. A fork turning round a pivot was used to connect the guard-ring to the projecting electrode of the protected disk, or the latter to earth through the galvanometer G.

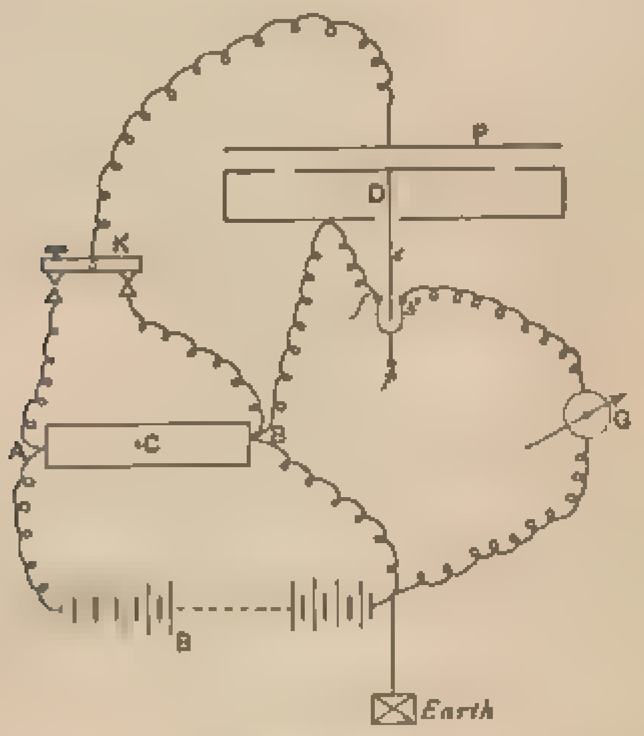

Fig. 140.

Arrangement of Apparatus.

The protected disk, D, of the condenser was a square of area of 1325·14 sq. cms., and was separated from the guard-ring by a gap 2·5 mm. wide. The distance between the plates was ·7728 cm. The plates were supported on well paraffined levelling screws of ebonite, and were strengthened by diagonal ribs on the upper side of the plate P, and the under side of the disk D.

* *Journ. Soc. Tel. Eng.* 1879.

The galvanometer was a Thomson's astatic instrument of about 20 ohms resistance. The ordinary needles were however replaced by small spheres each built up of a number of tiny magnets having their like poles all turned the same way, the spheres being completed with pieces of lead. The period T of the needle was 39·5 seconds, and its logarithmic decrement 1565.

Mode of Experimenting.

The mode of operating was as follows. The key K was depressed, and the plate P thereby connected to A; at the same time the electrode e was connected to f. Thus the condenser was charged to the difference of potential existing between A and B. Then the contact was broken between e and f, and the key released so as to make contact between P and B. This connected P and the guard-ring to earth while D was left insulated. The electrode e was then connected to g by the fork, and discharged the disk D through the galvanometer, the reading of which was observed.

Theory of Method.

The difference of potential E given by the battery between A and B was measured in the following manner. A very high resistance R was put in the circuit of the galvanometer, and its terminals were then connected to A and another point C in AB, enclosing between them a known fraction k of the whole resistance. The difference of potential between A and C was thus kE. The galvanometer was shunted through a resistance S, so that G being the resistance of the coil a current $kES\,\{R(G+S)+GS\}$ was sent through the instrument. The deflection thus produced was observed.

Now if θ and α denote the angular deflections given by the transient and the steady current respectively, and C_m the capacity in electro-magnetic units of the protected disk D, we have by the ballistic and tangent galvanometer formulæ

$$\frac{C_m E}{kES\,\{R(G+S)+GS\}} = \frac{T}{\pi}\,\frac{\sin\frac{1}{2}\theta}{\tan\alpha},$$

or

$$C_m = \frac{T}{\pi}\,\frac{kS}{R(G+S)+GS}\,\frac{\sin\frac{1}{2}\theta}{\tan\alpha}.$$

Thus C_s denoting the calculated capacity we find

$$v^2 = \frac{C_s}{C_m} = C_s\,\frac{\pi}{T}\,\frac{R(G+S)+GS}{kS}\,\frac{\tan\alpha}{\sin\frac{1}{2}\theta} \quad . \quad . \quad . \quad (22)$$

Final Results.

Three series of experiments were made consisting of 39, 41, and 18 discharges for T, 25·3, 39·5, 42·2 seconds respectively. The mean result obtained was

$$v = 2·98 \times 10^{10} \text{ in cms. per second.}$$

This however must be corrected for the value of the B.A. unit, and becomes

$$v = 2·955 \times 10^{10} \text{ in cms. per second.}$$

Klemencic's Experiments.

This method was used by Klemencic * with the modification that a rapid succession of discharges was sent through the galvanometer so that a constant deflection was produced. The mean result of two different researches by this method was

$$v = 3·041 \times 10^{10}$$

in cms. per second.

Similar experiments by Stoletow † gave

$$v \begin{array}{l} > 2·98 \times 10^{10} \\ < 3·00 \times 10^{10} \end{array}$$

in cms. per second.

Maxwell's Null Form of Method III.

The following form of the method, due to Maxwell,‡ has the advantage over that just described of being a null method, and therefore of not requiring any correction for torsion, damping, &c., while it shares with the former the advantage of involving the square root only of $\sqrt{C_s/C_m}$, and therefore only half of any error made in determining C_m or C_s. A Wheatstone bridge (Fig. 141) has a gap in one of the arms at p, q, and a contact piece or tongue, n, is made to vibrate across the gap so as to connect one plate of a condenser alternately to p and to q, while the other plate is kept permanently in contact with the point C. The resistances of the wires

* *Wien. Ber.* 83, 1881.

† *Soc. Franc. de Phys.* Nov. 4, 1881.

‡ *El. and Mag.* vol. ii. arts. 775, 776.

Cp, *qB* are made inappreciable, so that the plates of the condenser are alternately brought to the same potential, and charged to the potential existing between *C* and *B*.

A succession of transient currents are thus produced in the same direction through the galvanometer, and if *P*, *Q*, *S* are properly adjusted, are prevented by a steady current in the opposite direction from producing any deflection. From the condition, (29) below, fulfilled by the resistances of the bridge, the value of C_m can be

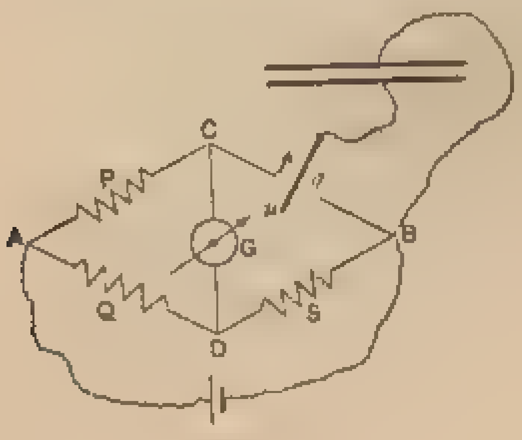

FIG. 141.

found, and compared as before with the value C_s of the capacity in electrostatic units.

So far as C_s is concerned the error of this method (and of others which require the capacity of a standard condenser) is only that involved in the measurement of the dimensions of the condenser, and reduces finally to that of the measurement of a length. Proper allowances can easily be made for want of accurate adjustment of the parts of the condenser. Accuracy of Method.

The determination of C_m is limited in accuracy only by the error involved in the use of the galvanometer, which must be so sensitive as to detect a sufficiently small variation of resistance. This error in the experiments described below was well within the limits of accuracy aimed at. In Thomson and Searle's investigations below it was estimated that the error from the galvanometer was not more than 1 in 2500 in the value of v.

Theory of Method.

Calling the resistances P, Q, S as marked on the figure, and denoting the currents from C to A, C to D, and B to A, by $\dot{x}$, $\dot{z}$, $\dot{u}$, the resistance and self-induction of the galvanometer by G and L, we have from the circuits $ACDA$, $ADBA$, the equations of currents, supposing all the branches, except CD, devoid of inductance,

$$\left.\begin{aligned} P(\dot{x} + \dot{z}) - Q(\dot{u} - \dot{x} - \dot{z}) + L\ddot{z} + G\dot{z} = 0 \\ Q(\dot{u} - \dot{x} - \dot{z}) + S(\dot{u} - \dot{x}) + B\dot{u} - E = 0 \end{aligned}\right\} \quad . \quad . \quad (23)$$

At the beginning and end of the charging of the condenser the currents have their steady values and therefore these equations become

$$P\dot{z}_0 - Q(\dot{u}_0 - \dot{z}_0) + G\dot{z}_0 = 0$$
$$Q(\dot{u}_0 - \dot{z}_0) + (B + S)\dot{u}_0 - E = 0$$

where the suffixes indicate the steady values of the currents.

Subtracting these last equations from the corresponding equations (23) for the variable state, and putting $\dot{u}_1$, $\dot{z}_1$, for $\dot{u} - \dot{u}_0$, $\dot{z} - \dot{z}_0$, we find

$$\left.\begin{aligned} P(\dot{z}_1 + \dot{x}) - Q(\dot{u}_1 - \dot{z}_1 - \dot{x}) + L\ddot{z} + G\dot{z}_1 = 0 \\ Q(\dot{u}_1 - \dot{z}_1 - \dot{x}) + S(\dot{u}_1 - \dot{x}) + B\dot{u}_1 = 0 \end{aligned}\right\} \quad . \quad (24)$$

The quantities $\dot{u}_1$, $\dot{z}_1$, it is to be noticed denote the excess in each case of the current flowing at any instant above the steady current, in consequence of the charging of the condenser, while $\dot{x}$ is the charging current.

Integrating, from the beginning of the charging to the end,

the equations just found, remembering that $\dot{z}$ has the value $\dot{z}_s$ at both limits, and rearranging, we get

$$\left.\begin{aligned}(P+Q)x + (G+P+Q)z_1 - Qu_1 &= 0\\ -(Q+S)x - Qz_1 + (Q+S+B)u_1 &= 0\end{aligned}\right\} \quad . \quad . \quad (25)$$

where x denotes the whole charge of the condenser, and u_1, z_1, the excess in each case of quantity of the electricity conducted by the currents $\dot{u}$, $\dot{z}$, above that which would have flowed in the same time if the current had remained constant. Theory of Method.

Eliminating u_1 from (25) we find

$$x = \frac{-(Q+S+B)(G+P+Q)+Q^2}{(P+Q)(Q+S+B)-Q(Q+S)}z_1 \quad . \quad . \quad (26)$$

But when the condenser is fully charged the difference of potential between its coatings is x/C_m, and this is $G\dot{z}_s + S\dot{u}_s$, so that

$$x = C_m(G\dot{z}_s + S\dot{u}_s).$$

Also clearly $(G+P)\dot{z}_s = Q(\dot{u}_s - \dot{z}_s)$, and therefore

$$\dot{u}_s = \frac{G+P+Q}{Q}\dot{z}_s$$

and

$$x = C_m\left(G + S\frac{G+P+Q}{Q}\right)\dot{z}_s. \quad . \quad . \quad . \quad (27)$$

If the condenser is charged and discharged n times a second, the quantity of electricity which passes through the galvanometer over and above that which passes in the steady current is nz_1. Hence, if there is no deflection, we must have $\dot{z}_s + nz_1 = 0$, or $\dot{z}_s = -nz_1$. Thus (27) becomes

$$x = -nC_m\left(G + S\frac{G+P+Q}{Q}\right)z_1. \quad . \quad . \quad . \quad (28)$$

This value of x used in (26) gives Value of C_m

$$nC_m = \frac{Q\{(Q+S+B)(G+P+Q)-Q^2\}}{\{P(Q+S+B)+QB\}\{S(G+P+Q)+GQ\}}. \quad (29)$$

If P and S are very great in comparison with the other resistances, this reduces to the approximate solution

$$nC_m = \frac{Q}{PS}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

The electromagnetic value of the capacity of the condenser having thus been found, that of v is of course obtained as before from the ratio $\sqrt{C_s/C_m}$.

The method has been carried out with this mode of determining C_m by Prof J. J. Thomson * in a very careful series of experiments giving the result

$$v = 2{\cdot}963 \times 10^{10}, \text{ in cms. per second,}$$

by Mr. E. B. Rosa † at Baltimore, and again by Prof J. J. Thomson and Mr. G. F. C. Searle ‡ at Cambridge in an elaborate research made with improved apparatus.

We shall describe here Mr. Rosa's experiments and the later investigation of Thomson and Searle.

Rosa's Experiments. Mr. Rosa used the standard spherical condenser described above as used by Prof. Rowland in his experiments on this subject.

The vibrating tongue u was operated by one or other of two forks made by Kœnig, of Paris, of frequencies 32 and 130 per second. These were maintained in vibration in the ordinary way by an electromagnet between the prongs worked by the current from three or four Bunsen cells.

Arrangement of Apparatus. With the slower fork a commutator was used, but with the faster fork a different arrangement was adopted. A wire led from the inner coating of the condenser was forked, and a branch of it connected by wax to the end face of each prong of the tuning-fork. The plane of vibration was vertical, and each wire was turned so as to dip into two mercury cups cut in fixed pieces of vulcanite, at a vertical distance apart equal to that between the prongs of the fork. The upper cup was connected with the point C of Fig. 141, the lower cup to B. Thus when the prongs moved apart the lower wire dipped into the mercury, connecting the inner ball of the condenser to B, while the upper broke con-

* *Phil Trans R.S.* 1883.
† *Phil Mag.* Oct. 1889.
‡ *Phil. Trans. R.S.* Vol. 181 (1890).

tact; when the prongs approached one another the upper contact was made and the lower broken, and the two plates of the condenser were put into direct contact. Thus in the former case the condenser was charged, in the latter discharged.

The galvanometer used was a very sensitive Thomson's astatic instrument.

The battery consisted of about 40 cells of a storage battery, giving an electromotive force of about 80 volts.

The resistances Q and S were taken from two resistance boxes by Elliott, containing 12,000 ohms and 100,000 ohms respectively.

Construction of High Resistances.

The resistance P, which was very great, was made by ruling pencil lines on ground glass, and protecting the surface of glass and graphite with a thick coat of shellac varnish. Connection was made at the ends by tinfoil pressed against the graphite by rubber packing. Ten such resistances were made and mounted in cylindrical cases, so that their temperatures might be maintained as nearly constant as possible. Their values were determined by a comparison (made by the method of Wheatstone's bridge with a ratio of about 100) with the resistances of the boxes used for Q and S, and proved very constant and reliable.

Adjustment of Condenser.

The capacity of the vibrating piece and the connecting wires was determined experimentally by separating them from the condenser. Special attention was given to the question as to whether the capacity of the charging wire might be taken as the same when the wire was in contact as when detached, and no appreciable difference was found.

The inner sphere was adjusted by lifting off the upper half of the outer shell, and adjusting the position of the ball relatively to the equatorial circumference of the shell, then replacing the hemisphere, and moving the ball vertically from contact at top to contact at bottom of the shell, and causing the contact in each case to be indicated by the closing of an electric circuit. The readings of a sliding vernier gave the top and bottom positions, and the mean of these readings the central position. It was estimated that the ball was centred to ·1 mm vertically and ·2 mm. horizontally, or to an error of less than 1 per cent. of the distance between ball and shell.

Now, for an eccentric cylinder, theory shows* that a similar displacement of 1 per cent. from centrality would give an error of capacity of 1/200 per cent., and a smaller error for a spherical

* J. J. Thomson, 'On the Determination of *v*.' *Phil. Trans. R.S.* 1883.

condenser. A displacement of four per cent., it was found by trial, caused a quite inappreciable change in capacity.

Measurement of Dimensions of Condenser.

The dimensions of the outer shell were determined by filling it with water and weighing, and of the inner ball by weighing it sunk in water by an attached mass, and making all necessary corrections for displaced air, &c. The results were checked by measurements made by callipers, compared with a standard metre bar. The results were :—

	Radius	
	By weighing.	By direct measurement.
Shell . . .	12·6805 cms.	12·6791 cms.
Ball A . . .	10·1180 „	10·1183 „
Ball B . . .	8·8735 „	8·8736 „

The experiments were made with the larger ball, and four series were made, the first, second, and fourth with both forks, the third with the slow fork alone.

Final Results.

It was found that the results for the fast fork were slightly lower than those for the slow fork, coming out according to the weights given to the observations,

$$v = 2{\cdot}9994 \times 10^{10} \text{ in cms. per second for the fast fork, and}$$

$$v = 3{\cdot}0023 \times 10^{10} \text{ in cms. per second for the slow fork.}$$

The results for the fast fork were the more uniform and it was thought the more accurate, and were given double weight in striking the final mean. Thus the final result of all the experiments was

$$v = 3{\cdot}0004 \times 10^{10} \text{ in cms. per second.}$$

The results of Series II. and III. were greater than those of I. and IV., and it was thought possible that the halves of the outer shell had been very slightly separated in the former case by an obstruction in the flange of junction. It is to be noticed that

the results with the slow fork are the greater, indicating too *small* a value of C_m. This is the kind of result which the *fast* fork might be expected to give if the period was not long enough to allow the condenser to be fully charged. The rejection of the observations of Series II. and III. would give

$$v = 2{\cdot}9993 \times 10^{10} \text{ in cms. per second,}$$

which only differs from the former value by $\frac{1}{30}$ per cent.

Fig. 142.

Thomson and Searle's Experiments.

In Thomson and Searle's investigation the condenser used was cylindrical, and was provided with a guard-ring at top and bottom, so that the effect of the ends was in great measure avoided. The condenser is shown in section in Fig. 142. The

Arrangement and Measurement of Condenser.

dimensions of the inner cylinder were measured by accurate callipers in the most careful manner. It was found that the cylinder was slightly elliptic in section, as shown in the following statement of results of measurement:—

Top end : maximum diameter	. .	23·5302 cms.
,, minimum ,,	. .	23·5161 ,,
Bottom end : maximum diameter	.	23·5348 ,,
,, minimum ,,	.	23·5169 ,,
	Mean	23·5245

The internal diameter of the outer cylinder was measured by callipers specially provided for this purpose with projecting steel pieces on their jaws. The results obtained for two diameters at right angles to one another at each end of the cylinder gave a mean diameter of 25·4114 cms.

The internal cylinder was supported on pieces of ebonite placed on the lower ring, and the upper ring on similar pieces on the internal cylinder. The outer cylinder was also in three parts, two ring pieces for top and bottom, and a long central piece corresponding to the internal cylinder.

Measurement of Dimensions of Condenser. Correction for Guard-ring Gap.

The length of the internal cylinder was measured by applying the jaws of a beam compass to its ends and measuring under microscopes first the distance between two marks, one on each jaw, then the distance between these marks when the jaws were put close.

The length of the cylinder was found to be 60·9784 cms. The correction for want of equality in the distribution caused by the two equal air spaces was calculated and found to amount, within 1 part in 2000, to a lengthening of the internal cylinder by the breadth of one air-space. The mean allowance for the gaps at the guard-ring was thus found to be ·2907 cm., so that the total effective length of the internal cylinder was 61·2691 cms.

The distance between the inner and outer cylinders was determined by fastening down the internal cylinder, and the outer cylinder of the same length, in co-axial position on a glass plate with cement, and fixing a glass cover on top; then filling, by means of two openings left in the cover, the annular space between the cylinders with water. The water was taken from a flask containing a known weight of water, and so by a second weighing of the flask the weight of water used was obtained. The weighings were all corrected to vacuum, and for error in weights, effect of temperature, &c.

The volume was found to be 4412·08 cubic cms., so that the mean distance d between the cylinders was, with the radii given above, ·94128 cm. The ratio of external and internal radii a, b used was thus $1 + \cdot 94128/11 \cdot 76225 = 1 \cdot 0800262$. Thus

Electrostatic Value of Capacity.

$$C_s = \frac{l}{2 \log \frac{a}{b}} = \frac{61 \cdot 2691}{\cdot 15397063} = 397 \cdot 927$$

in centimetres.

The measurement of capacity in electromagnetic units was made by the method already described, somewhat modified on account of the existence of the guard-ring. The arrangement of apparatus is shown in Fig. 143. The condenser plate is shown

Determination of Electromagnetic Value of Capacity.

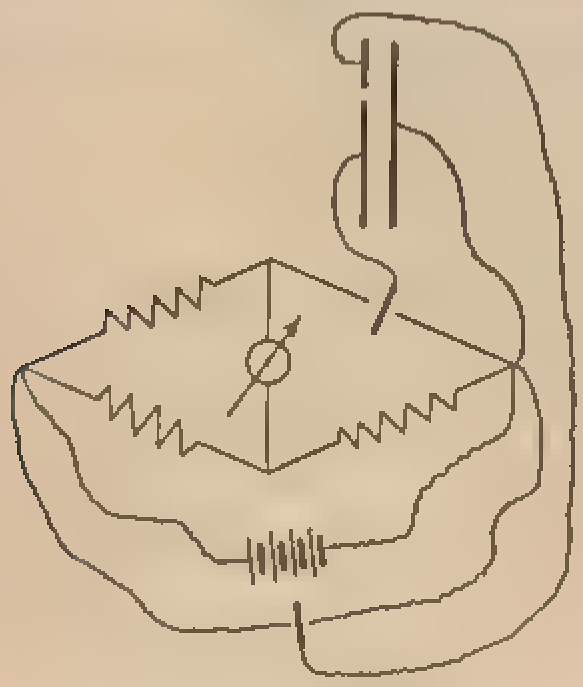

FIG. 143.

connected as before to a contact-making piece u, which makes contact alternately with p and q, while one guard-ring is connected with a second contact-piece v, which makes contact alternately with r and s. The pieces p and q represent the contact-plates of a commutator which alternately came into contact with a spring or brush, u, connected with the inner coating of the condenser; r and s represent the contact-plates of another commutator, v a brush which alternately connected them with the guard-ring.

Action of Commutators. The two commutators were mounted on the same axis, so that they were kept always in the same relative position. When the commutators were worked the following contents were made in the order indicated by the numbers. V_A, V_B, V_C, denote the potentials of the points A, B, C, respectively.

1. $\left\{\begin{array}{l} u \text{ on } q: \text{ condenser discharged.} \\ v \text{ on } s: \text{ guard-ring discharged.} \end{array}\right.$

2. $\left\{\begin{array}{l} u \text{ on } p: \text{ condenser begins to charge.} \\ v \text{ on } s. \end{array}\right.$

3. $\left\{\begin{array}{l} u \text{ on } p: \text{ condenser charged to potential } V_A - V_B. \\ v \text{ on } r: \text{ guard-ring charged to potential } V_C - V_B. \end{array}\right.$

4. $\left\{\begin{array}{l} u \text{ on } q: \text{ condenser begins to discharge.} \\ v \text{ on } r. \end{array}\right.$

5. $\left\{\begin{array}{l} u \text{ on } q: \text{ condenser discharged.} \\ v \text{ on } s: \text{ guard-ring discharged.} \end{array}\right.$

Theory of Method. According to the notation already adopted above we denote the currents in Cp, CD, BA, by $\dot{x}$, $\dot{z}$, $\dot{u}$; in addition, in the present case we have, when v is in contact with r, a current in Ar. Let this be denoted by $\dot{w}$. The circuits $ACDA$, $ADBA$, give the equations

$$\left.\begin{array}{l} -Q(\dot{u}-\dot{x}-\dot{z}-\dot{w}) + P(\dot{x}+\dot{z}) + L\ddot{z} + G\dot{z} = 0 \\ Q(\dot{u}-\dot{x}-\dot{z}-\dot{w}) + S(\dot{u}-\dot{x}-\dot{w}) + B\dot{u} - E = 0 \end{array}\right\} \quad . \quad . \quad (31)$$

At the beginning and end of the charging the currents have their steady values, and then as at p. 626,

$$\left.\begin{array}{l} -Q(\dot{u}_0-\dot{z}_0) + P\dot{z}_0 + G\dot{z}_0 = 0 \\ Q(\dot{u}_0-\dot{z}_0) + (B+S)\dot{u}_0 - E = 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad (32)$$

These subtracted from the corresponding pair of equations (31) for the varying state, give, if $\dot{u}_1$, $\dot{z}_1$ denote $\dot{u}-\dot{u}_0$, $\dot{z}-\dot{z}_0$, respectively,

$$\left.\begin{array}{l} -Q(\dot{u}_1-\dot{z}_1-\dot{x}-\dot{w}) + P(\dot{x}+\dot{z}_1) + L\ddot{z} + G\dot{z}_1 = 0 \\ Q(\dot{u}_1-\dot{z}_1-\dot{x}-\dot{w}) + S(\dot{u}_1-\dot{x}-\dot{w}) + B\dot{u}_1 \quad = 0 \end{array}\right\} \quad . \quad (33)$$

These integrated from the beginning of the charging to the end yield

$$\left.\begin{aligned}(P+Q)x+(G+P+Q)z_1+Qw-Qu_1&=0\\-(Q+S)x-Qz_1-(Q+S)w+(Q+S+B)u_1&=0\end{aligned}\right\}\quad(34)$$

where x, as before, denotes the whole charge of the inner coating of the condenser, while w denotes that of the guard-ring.

Elimination of u_1 from (34) gives

$$\{P(Q+S+B)+BQ\}x+BQw$$
$$=-\{(G+P+Q)(Q+S+B)-Q^2\}z_1 \quad . \quad . \quad (35)$$

This differs from the former equation (26) only in having the term BQw on the left.

When the condenser is fully charged we have as before

$$x=C_m\left(G+S\frac{G+P+Q}{Q}\right)i_s \quad . \quad . \quad . \quad . \quad (36)$$

and further if C'_m be the capacity of the guard-ring

$$w=C'_m\left(G+P+S\frac{G+P+Q}{Q}\right)i_s \quad . \quad . \quad . \quad (37)$$

since the multiplier of C'_m on the right is the final difference of potential between A and B.

Again if there be no galvanometer deflection $i_s+nz_1=0$, or $i_s=-nz_1$, so that (36) and (37) become

$$\left.\begin{aligned}x&=-nC_m\left(G+S\frac{G+P+Q}{Q}\right)z_1\\w&=-nC'_m\left(G+P+S\frac{G+P+Q}{Q}\right)z_1\end{aligned}\right\}\quad . \quad . \quad (38)$$

These substituted in (35) give

Value of C_m in Terms of Resistances of Bridge.

$$nC_m\{P(Q+S+B)+BQ\}\{S(G+P+Q)+GQ\}$$
$$+nC'_mBQ\{(G+P)Q+S(G+P+Q)\}$$
$$=Q\{(G+P+Q)(Q+S+B)-Q^2\} \quad . \quad (39)$$

Correction for Difference of Potential between Guard-ring and Protected Cylinder.

The second term on the left was negligible in the experiments made, inasmuch as the resistance B of the battery was small in comparison with the other resistances. Thus the value of C_m was given as before by (29). It was necessary to apply a correction for the small difference of potential δV between the guard-ring and the inner cylinder after charging, which prevented the distribution on the inner cylinder from being so nearly uniform as it otherwise would have been. It is shown in the paper that this correction could be made by adding to the internal cylinder a strip of breadth

$$h\left(\frac{t}{c} - \frac{2}{\pi}\log\frac{4c}{h\epsilon}\right)\frac{\delta V}{V},$$

where V is the difference of potential between the cylinders, t the thickness of the guard-ring, c the half thickness of the pieces of ebonite supporting the guard-ring, h the distance between the cylinders, and ϵ the base of the Napierian system of logarithms. The coefficient of $\delta V/V$ was approximately 7·5, and from the values given above

$$\delta V = -\left\{G + S\frac{G+P+Q}{Q} - G - P - S\frac{G+P+Q}{Q}\right\}t_s = Pt_s,$$

$$V = \left(G + S\frac{G+P+Q}{Q}\right)t_s;$$

so that

$$\frac{\delta V}{V} = \frac{1}{183} \text{ nearly.}$$

Thus the correction was a strip of breadth 7 5/183 cm., or about 1 part in 1800 of the whole.

Description of Commutators.

Each commutator consisted of two rings with projecting semi-cylindrical pieces overlapping, as shown in Fig. 144, mounted on an ebonite casing round the common axis.

Two springs, shown in Fig. 145, made permanent contact with grooves in the ring portions of the contact-pieces, and formed the connections to the points CA and AB of the bridge. The charging contacts on the commutator were made with a brush of fine brass wire. On the axle are fixed the driving pulleys and a stroboscopic disk for the observation of the speed, by means of a maintained fork in the manner already sufficiently

described at p. 569 above. A side view of the stroboscopic disk is shown on the right in Fig. 144.

The worm-wheel and endless screw were used to make a contact with a spring at every revolution of the wheel, that is every 30 turns of the commutator, to excite one of the electromagnets of the recording apparatus referred to below. The commutator was driven by a water-motor and long cord made of fishing-line joined in a long splice to prevent inequalities in speed. The speed was regulated by letting the cord run through the fingers.

The stroboscopic disk, Fig. 144, had, as shown, five circles

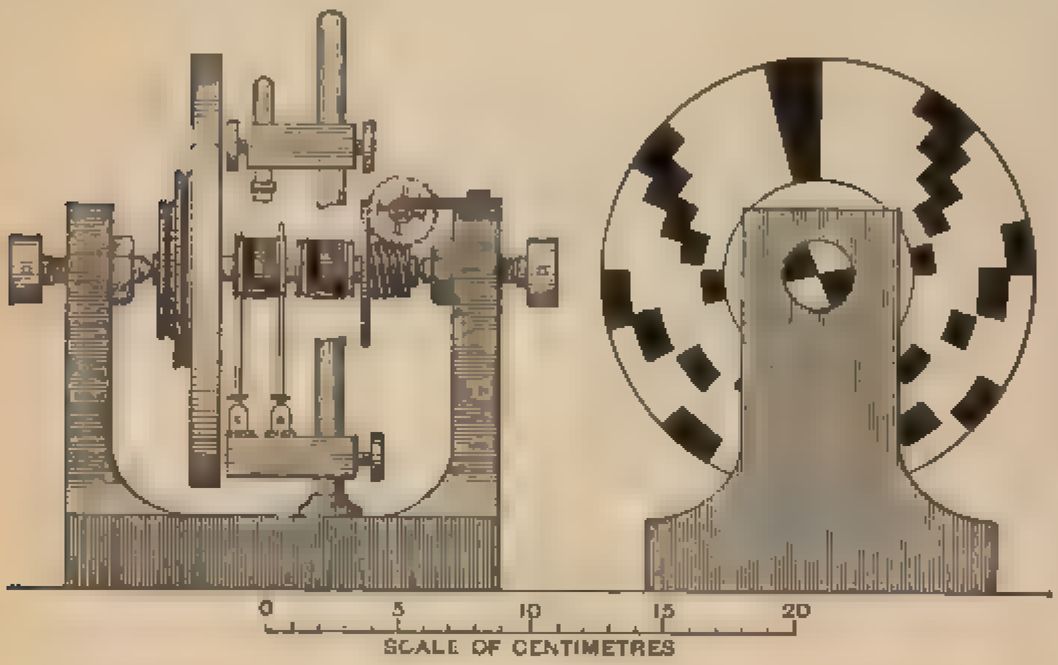

FIG. 144.

containing 4, 5, 6, 7, 8 black spots at equal intervals; the fork making 64 complete vibrations per second, and the commutator not running much faster than 80 revolutions, the speeds of the disk from 16 revolutions per second upwards when a stationary pattern was visible were the fractions Strobo-scopic Disk.

$$\tfrac{1}{4}, \tfrac{2}{7}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{4}{7}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{6}{7}, 1, \tfrac{8}{7}, \tfrac{6}{5}, \tfrac{5}{4}, \tfrac{4}{3}, \tfrac{4}{3},$$

of 64 revolutions per second.

The electrically driven fork maintained another of about twice its frequency, and the latter gave beats with Lord Rayleigh's

standard fork, so that the speed of the observing fork was obtained.

Determination of Frequency of Standard Fork. The frequency of the standard fork was redetermined by causing the worm-wheel driven by the commutator to make a mark on a running tape every 30 revolutions of the commutator. This was effected by the completion of a circuit which excited an electromagnet, and thereby caused an armature to descend slightly, and bring an inked roller down on the paper. A mark was similarly made on the tape every second by the completion of a circuit by the laboratory clock. Fig. 146 shows the electromagnets, armature, and marking roller, with an inking drum above, on which the roller made contact when the armature was not pulled down.

The method of experimenting was as follows.

The beats between standard and auxiliary forks were counted.

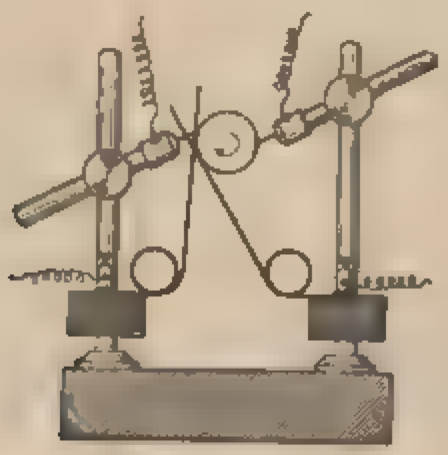

Fig. 145.

Mode of Experimenting The motor was then started and the commutator kept at a constant speed by the disk, and after the apparatus was stopped the beats were again counted. Thus the speed of the observing fork was directly measured, and that of the standard obtained from the beats. Three observations gave a mean of 128·1045 for the frequency at 16° C., a slightly smaller frequency than that found by Lord Rayleigh. The difference was attributed to secular softening of the steel in the intervening six or seven years.

The resistances were taken from resistance boxes which were carefully compared with standard coils.

The galvanometer had a resistance of 17380 ohms, and had two coils of about 16000 turns each. The coils were very care-

fully insulated, and showed no leakage when tested by a gold-leaf electroscope.

The current was produced with 36 small storage cells, arranged in two parallels of 18 cells each. It was also carefully insulated.

All the quantities observed were corrected with great care for

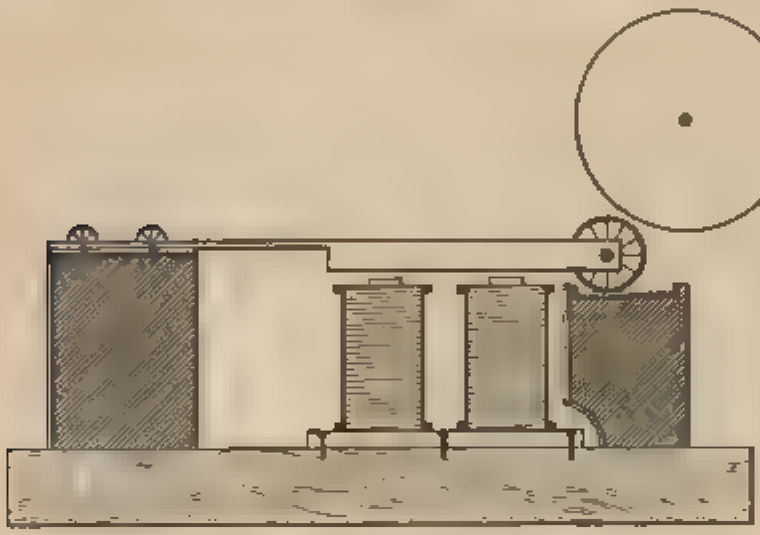

FIG. 146.

temperature variations, and the capacity of the connecting wires to the condenser was taken into account.

Three sets of experiments 7, 10, and 6 in number were taken, and gave as mean values of C_m, $443{\cdot}471 \times 10^{-21}$, $443{\cdot}417 \times 10^{-21}$, $443{\cdot}569 \times 10^{-21}$ C.G.S.; or as mean of all $C_m = 443{\cdot}486 \times 10^{-21}$ C.G.S. electromagnetic units. Thus

$$v = \sqrt{\frac{397{\cdot}927}{443{\cdot}486 \times 10^{-21}}} = 2{\cdot}9955 \times 10^{10},$$ Final Result.

in cms. per second.

Methods of comparing the capacity C_m of a condenser with the self-inductance L of a coil have been given above, Chap. VIII. If then the capacity of a condenser has been thus found, in terms of a self-inductance L which can be exactly calculated, the value C_s in electrostatic units can be found either directly by calculation for the condenser, or, if that is not possible, by comparison with the accurately known capacity of a standard condenser. Method IV. Determination of C_m by Comparison with Inductance.

Thus if, as at p. 491 above,

$$C = \frac{L}{QR}$$

we have

$$v = \sqrt{\frac{C_s}{C_m}} = \sqrt{\frac{C_sQR}{L}} \quad . \quad . \quad . \quad . \quad (40)$$

Method V. Determination of Product of a Capacity and an Inductance. The next two methods are mainly of theoretical interest. According to Method V., a magnet is rotated within a coil suspended with its plane vertical by a bifilar. The current induced in the coil causes it to turn round a vertical axis, and, if the period of rotation be constant and small in comparison with the period of vibration, to take up a constant deflection. The coil is in circuit with a fixed coil of considerable self-inductance, so that the whole inductance of the circuit is L, and with a condenser of capacity C. The value of CL can be found by observing the deflections D_1, D_2, D_3, for three different angular velocities n_1, n_2, n_3, of the magnet. Then

$$C^2L^2 = \frac{1}{n_1n_2n_3} \frac{\sum \frac{n_1^3}{D_1}(n_2^2 - n_3^2)}{\sum \frac{n_1}{D_1}(n_2^2 - n_3^2)} \quad . \quad . \quad . \quad . \quad (41)$$

If the induction through the coil due to the magnet when its axis is parallel to that of the coil be M, then when the magnet has turned through the angle θ from that position the induction is $M \cos \theta$, or $M \cos nt$, if n denote the angular velocity, and t be reckoned from the instant at which $\theta = 0$.

Theory of Method. If x be the whole quantity of electricity which has flowed through the circuit from the era of reckoning, the current is $\dot{x}$, and the induction through the circuit due to the current in it is $L\dot{x}$. Thus if E denote the difference of potential between the plates of the condenser, the electromotive force producing the current is $E + d(L\dot{x} + M \cos nt)/dt$, and the equation of currents is

$$R\dot{x} + \frac{d}{dt}(L\dot{x} + M \cos nt) + E = 0.$$

But $CE = x$, so that this equation becomes

$$CL\frac{d^2x}{dt^2} + CR\frac{dx}{dt} + x = CMn \sin nt \quad . \quad . \quad . \quad (42)$$

From (42) it is clear that the values of CL and CR are the same whether the electromagnetic or the electrostatic system of units is used.

This differential equation is one of forced oscillation, so that for x we have the equation

$$x = -\frac{MCn}{\sqrt{R^2C^2n^2 + (1 - CLn^2)^2}} \cos(nt - e) \quad . \quad . \quad (43)$$

where

$$\tan e = -\frac{1 - CLn^2}{RCn}.$$

The couple on the suspended coil produced by electromagnetic action is at time t

$$\Theta = \dot{x}nM \sin nt,$$

and the mean value Θ of this over one revolution is, since $2\pi/n$ is the period,

$$\bar{\Theta} = -\frac{n}{2\pi} \frac{M^2Cn^2}{\sqrt{R^2C^2n^2 + (1 - CLn^2)^2}} \int_0^{2\pi/n} \sin(nt - e) \sin nt \, . \, dt$$

$$= -\frac{1}{2} \frac{n^3M^2C^2R}{R^2C^2n^2 + (1 - CLn^2)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

If the coil have a sufficiently great moment of inertia the variations of the couple acting on it will not cause it to oscillate sensibly, but it will take up a position of equilibrium depending on the mean couple $\bar{\Theta}$.

The mean deflection D is proportional to $\bar{\Theta}$ and so

$$P\frac{n}{D} = R^2C^2 + \left(\frac{1}{n} - CLn\right)^2 \quad . \quad . \quad . \quad . \quad (45)$$

where P is a constant. By means of three different angular velocities three equations of this form are obtained, which give (41) by elimination of P and R.

If the experiment were carried out it would be desirable to take say n_2 as that for which n/D is a minimum, that is $n_2^2 = 1/CL$, and n_1, n_3, one greater, the other less than n_2.

Since $v^2 = C_s/C_m$, we have, if L_s, L_m denote the electrostatic and electromagnetic values of L,

$$C_s L_m = v^2 C_s L_s = v^2 C_m L_m.$$

Hence

$$v^2 = \frac{C_s L_m}{C_m L_m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

The denominator of the expression on the right is determined experimentally, as explained above, and the numerator is obtained by direct calculation of C_s and L_m, or by comparison of the condenser and circuit with proper standards.

Method VI. By Electrostatic Measurement of High Resistance.

Method VI. involves the determination of the electrostatic value, R_s, of a high resistance, through which a condenser of capacity C_s is discharged. This can be done by measuring the rate of fall of difference of potential between the plates of the condenser by means of an electrometer connected with them. If V be the electrostatic value of the difference of potential at any time t we have

$$C_s \frac{dV}{dt} + \frac{V}{R_s} = 0$$

and therefore

$$\log V + \frac{t}{C_s} = A$$

where A is a constant. If V be the difference of potential t seconds after it was V_0, we get from this equation

$$\frac{t}{C_s R_s} = \log \frac{V_0}{V}$$

or

$$R_s = \frac{t}{C_s} \frac{1}{\log \frac{V_0}{V}}.$$

If $V = \frac{1}{2} V_0$, $R_s = t/C_s \log 2$.

If now R_m is known we have, since $C_s R_s = C_m R_m$, $R_m/R_s = C_s/C_m = v^2$, and therefore

$$v^2 = \frac{R_m C_s \log 2}{t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

The method of electrical oscillations has been used by Lodge and Glazebrook.* An air condenser was made to discharge

* B. A. Report, 1889, or *Electrician*, vol. 23 (1889), p. 544.

through a coil of measurable inductance across a spark-gap between a pair of knobs about a millimetre apart. The condenser consisted of 11 squares (each 2 feet in side) of plate glass silvered on both sides, set up parallel to one another with a distance of 5 mms. between each pair of opposed silvered surfaces, and the silvered surfaces of the alternate plates joined metallically to form the coatings of the condenser. It had thus a capacity of about 600 metres in electrostatic measure. The coil was composed of about three miles of india-rubber covered wire of No 22 gauge, and had diameters 19 inches and 11 inches, and thickness 4 inches. Its inductance was about $4{\cdot}5 \times 10^9$ cms. in electromagnetic measure.

Method VII. By Measurement of Period of Electrical Oscillations. Lodge and Glazebrook's Experiments.

The condenser was charged by a Voss machine arranged to give a brush discharge across half an inch of air to the inner coating, while the other coating (that is, the two outer plates and the four alternate interior plates) were connected to earth.

The sparks were photographed on a revolving sensitive plate on which the knobs were focused by a quartz lens. The plate was driven by a water motor at a speed of about 64 turns per second, and its speed measured as in Lord Rayleigh's determination of the ohm, by observation of a stroboscopic disk through a slit alternately opened and closed by the vibration of an electrically maintained tuning-fork. The result was that a pattern was produced on the plate consisting of a long band, with a bead-like broadening for each half-oscillation. The period of vibration was thus measured with great exactness.

The resistance and inductance of the circuit could also be obtained with very considerable accuracy, as the resistance of the spark-gap was inappreciable. The value of L for the coil could also be obtained by direct calculation or by comparison with another coil.

The value of the period given above (p. 188) furnishes for these data the electromagnetic value of the capacity of the condenser. (See also Chap. XIV. below.) Also C_s can be found from an exact comparison with a standard condenser, and thus v can be obtained by (21) above.

The final results of the experiment do not seem yet (July, 1892) to have been published.

General Table of Results.

The following table gives the values of v obtained by different experimenters, and for comparing the velocity of light as determined experimentally by the methods of Fizeau and Foucault. It is in great part taken from Mr. E. B. Rosa's paper already referred to. The various results given were corrected by Rosa where necessary to the value ·98664 ohm for the B.A. unit:

Table of Values of v.

Ratio of Units			Velocity of Light.		
Date.	Experimenter	V in cms. per sec	Date.	Experimenter	Vel of Light in cms. per sec
1856 [1]	Weber and Kohlrausch ...	3·107 × 10^{10}	1874	Cornu— (Method of Fizeau) ...	2·9850 × 10^{10}
1868 [2]	Maxwell	2·842 × 10^{10}	1874	" "	3·0040 × 10^{10}
1869 [3]	W Thomson and King ...	2·808 × 10^{10}	1880–1	Young and Forbes (Method of Fizeau, modified by Forbes) ...	3·0138 × 10^{10}
1872 [4]	McKichan	2·896 × 10^{10}	1879	Michelson— (Method of Foucault) ...	2·9991 × 10^{10}
1879 [5]	Ayrton and Perry	2·960 × 10^{10}	1882	" "	2·9985 × 10^{10}
1880 [6]	Shida	2·955 × 10^{10}	1882	Newcomb— (Method of Foucault) ...	{ 2·9986 × 10^{10} 2·9981 × 10^{10}
1881 [7]	Stoletow	2·99 × 10^{10}			
1882 [8]	F Exner	2·92 × 10^{10}			
1884 [9]	Klemencic	3·019 × 10^{10}			
1883 [10]	J J Thomson	2·963 × 10^{10}			
1888 [11]	Himstedt	3·009 × 10^{10}			
1889 [12]	Rowland	2·9815 × 10^{10}			
1889 [13]	E B Rosa	3·0004 × 10^{10}			
1889	W Thomson	3·004 × 10^{10}			
1890 [14]	J J Thomson and Searle	2·9955 × 10^{10}			

[1] *Pogg Ann* 1856.
[2] *Phil Trans R S* 68
or *Rep of Papers*, 2. p. 125.
[3] *B A Report* 1869
[4] *Phil. Trans R S* 1872.
[5] *Jour Soc. Tel. Eng* 1879
[6] *Phil Mag* 10 1880
[7] *Soc Franc de Phys* Nov 1881.
[8] *Wien. Ber* 86, 1882.
[9] *Wien. Ber.* 89, 1884
[10] *Phil Trans R S* 1883
[11] *Wied Ann* No 9 1888
[12] [13] } *Phil Mag* Oct 1889.
[14] *Phil. Trans R S.* 181 (1890).

CHAPTER XII

THE MEASUREMENT OF ACTIVITY IN ELECTRIC CIRCUITS

Activity in Circuit of Generator and Motor.

When a circuit in which a current of electricity is flowing contains a motor, or machine by which work is done in virtue of electromagnetic action, the whole electrical work done in the circuit consists, as was first shown by Joule, of two parts, work spent in heat in the generator and motor and in the conductors connecting them, and work done in moving the motor against external resistance. The total rate at which electrical energy is given out in the circuit is as we have seen, $E\gamma$ watts, where E is the total electromotive force of the generator in volts, and γ is the number of amperes of current flowing. The rate at which work is spent in heat is in watts, by Joule's law, $\gamma^2 R$, where R is the total resistance in circuit in ohms; hence, if we call W the rate at which work is done in the motor,* we have,

$$E\gamma = \gamma^2 R + W \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We may write this equation in the form,

$$\gamma = \frac{E - W/\gamma}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* We consider here a system in which γ is constant, and neglect loss of energy due to local currents, &c., in the motor. For information regarding the construction of practical motors and their action see a paper by Profs. Ayrton and Perry, *Proc. Soc. Tel. Engs.*, 1883, republished in the electrical journals, Kapp's *Electric Transmission of Power*, and Prof. S. P. Thompson's *Dynamo-Electric Machinery*.

which shows that the current flowing is equal to that which would flow in the circuit if, the resistance remaining the same, the motor were held at rest, and the electromotive force diminished by an amount equal to W/γ. This is what is called the *back electromotive force* of the motor, and is due to the action of the motor in setting up an electromotive force tending to send a current through the circuit in the opposite direction to that of the current by which the motor is driven. We shall denote the back electromotive force by E_1. Hence equation (2) becomes,

Back E. M. F. of Motor.

$$\gamma = \frac{E - E_1}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the rate at which work is spent in driving the motor is $E_1\gamma$.

To determine E we have simply to measure with a potential galvanometer or voltmeter, the difference of potential between the two terminals of the generator. Calling this V, and R_1 the effective resistance of the generator, we have plainly,

$$E = V + \gamma R_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Again, since γ and also the total resistance R in the circuit can be found by measurement, we find by (3)

$$E_1 = E - \gamma R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where all the quantities on the right-hand side are known.

Electrical Efficiency of Arrangement of Generator and Motor.

The ratio of $E_1\gamma$, the electrical energy spent per unit of time in the circuit otherwise than in heating the conductors, to the whole electrical energy $E\gamma$ spent in the circuit per unit of time, that is the ratio of E_1 to E, we may call the electrical efficiency of the arrangement. Denoting this efficiency by e, we find, by equation (4),

$$e = \frac{E_1}{E} = 1 - \frac{\gamma R}{E} = 1 - \frac{E - E_1}{E} \quad . \quad . \quad (6)$$

Arrangement of Maximum Electrical Efficiency.

Hence the smaller γ is made, that is, the slower the energy is given out, the value of the efficiency of the arrangement is the more nearly equal to *unity*, the value of the efficiency of an arrangement in which the energy in the motor done against external resistance is exactly equal to the whole electrical energy given out in the circuit.

When however energy is spent at the maximum rate in working the motor, $E_1\gamma$ has its greatest value. But by (5)

$$E_1\gamma = E\gamma - \gamma^2 R = W.$$

This equation may be written,

$$\gamma^2 R - E\gamma + W = 0,$$

a quadratic equation of which the solution is,

$$\gamma = \frac{E \pm \sqrt{E^2 - 4RW}}{2R}.$$

Now in order that these values of γ may be *real*, $4RW$ cannot be greater than E^2. Hence the greatest value W can have is $E^2/4R$. When W has this maximum

Arrangement of Maximum Activity.

value, γ is equal to $E/2R$, and therefore E_1 equal to $E/2$. Hence the electrical efficiency is $\frac{1}{2}$. It is to be very carefully observed that although in this case the arrangement is that of *greatest electrical activity*, it is *not that of greatest electrical efficiency*, as it has only about one-half the efficiency of one in which energy is given out at a very slow rate. The case is exactly analogous to that of a battery arranged so as to give maximum current through a given external resistance (see Vol. I. p. 148).

All that has been stated above is applicable to the case of a motor fed by any kind of generator whatever. The generator employed however is generally some form of dynamo- or magneto-electric machine driven by an external motor, such as a steam- or gas-engine or a water-wheel, and a few of the results obtained below apply only to such cases, which will be indicated as they occur.

Case of Generator and Motor Similar Machines.

When the generator and motor are exactly similar machines, and the same current passes through both, the ratio of E_1 to E will be that of $nAf(\gamma)$ to $n'Af(\gamma)$; where n and n' are the speeds of the machines, A a constant depending on the form and disposition of the magnets, and $f(\gamma)$ a function of the current. Hence in this case the efficiency is measured simply by the ratio of the speed of the motor to that of the dynamo. The more nearly therefore the speed of the motor approaches to that of the generator, the greater is the efficiency. It is to be observed however that two machines identically alike will not in practice be perfectly similar in their action, even with the same

currents flowing in their armatures and field-magnet coils. The armature currents tend to weaken the field in the generator, and to strengthen the field in the motor.

In general, the higher the speed at which the motor is run, the greater is the electrical efficiency of any arrangement, for it is obvious that the higher the speed the more nearly does E_1 approach to E, and therefore the value of E_1/E, the measure of efficiency, to unity.

For a constant difference $E - E_1$, the ratio of the energy spent in heating the conductors by the current to the whole energy expended in the circuit, may be reduced by increasing the electromotive force E of the circuit. If E be increased to nE while E_1 is changed to E'_1, so that $nE - E'_1 = E - E_1$, the electrical efficiency is raised to $(n-1)/n + E_1/nE$, or $\{(n-1)/n + 1/n^{th}\}$ of the former efficiency. Clearly as n is increased this approaches more and more nearly to unity.

Electrical Efficiency increased by increasing E.M.F. in Circuit.

The energy spent in heat is $\gamma^2 R$, or $(E - E_1)^2/R$, and the ratio of this to $E\gamma$ is $\gamma R/E$. But γR is equal to the constant difference $E - E_1$, hence the ratio is $(E - E_1)/E$, and this becomes smaller as E is increased. A greater efficiency is therefore obtained by using high potentials than by using low potentials. Hence a greater electrical efficiency is realized, with a given magneto- or dynamo-electric machine used as generator and a given motor, when both generator and motor are run at higher speeds. Consequently the generator should be run as fast as possible, and the motor loaded lightly, or the speed with which the working resistance is overcome reduced by gearing between it and the motor.

Effect of increased E.M.F. in different Cases.

When high potentials are obtained by the use of machines wound with fine wire, or by using as generator a battery of a large number of cells joined in series to drive a high potential motor, the gain of electromotive force is accompanied by an increase of resistance in the circuit. But if we suppose the

speed of the motor to be so regulated that the difference between the total electromotive force in the circuit and the back electromotive force of the motor remains the same in the different cases, it is easy to show that the electrical efficiency of the arrangement is greater for high electromotive forces than for low. If, as supposed, $E - E_1$ remains constant, while E is changed to nE, we have for the total activity of the motor $nE\gamma - (E - E_1)\gamma$. Dividing this by $nE\gamma$ we get for the electrical efficiency,

$$e = \frac{n-1}{n} + \frac{1}{n}\frac{E_1}{E} \quad \ldots \ldots \ldots \quad (7)$$

As n is made greater and greater, the first term on the right becomes more and more nearly equal to unity, and the last term to zero. Hence, on the supposition made, the efficiency is increased by increasing the working electromotive forces. Taking as a particular case $n = 2$, we see that the efficiency is $\frac{1}{2}$ together with one-half of the former efficiency; if $n = 4$, the efficiency is $\frac{3}{4}$ together with one-fourth of the former efficiency, and so on for other values of n. This result holds for any case whatever in which the condition that $E - E_1$ should remain constant is fulfilled; and hence it is independent of any change that may have been made in the resistances of the generator or motor in order to obtain the higher electromotive force nE. For example, it is plain that no sensible change in the actual rate of loss by heating of the conductors by the current will be produced by increasing the resistances of the generator and motor, if these be very small in comparison with the remainder of the resistance in circuit, as, since $E - E_1$ remains constant and the resistance is practically the same as before, the current strength will not be perceptibly altered. The ratio, however, of the activity wasted in heating to the total activity will be only $1/n$th of what it was before. In the opposite extreme case, in which the generator and motor have practically all the resistance in circuit, the current, $\gamma\,(=(E - E_1)/R)$, is diminished in the ratio in which the resistance is increased; and the actual rate of loss by heat according to Joule's law, $(E - E_1)^2/R$, is diminished in the same ratio, so that, as in the former case, its ratio to the total activity $nE\gamma$ is $1/n$th of what it was for the electromotive force E. We see, therefore, that here also the efficiency must be the same in both cases.

Working Efficiency of Motor.

We have called E_1/E *the electrical efficiency of the arrangement*, but this is not to be confounded with the efficiency of the motor itself. The activity $E_1\gamma$ includes the wasted activity, or rate at which work is done against frictional resistances in the motor itself, and in the gearing which connects it with its load, as well as the useful activity or rate at which it performs useful work. Hence, although the electrical efficiency of the arrangement be very great, only a small amount comparatively of the energy given to the motor may be usefully expended, and *vice versâ*; and we define therefore the efficiency of a motor at any given speed as the ratio of the useful activity to the whole activity, taking as the latter the total rate at which electrical energy is expended in the motor; that is, $E_1\gamma + \gamma^2 R_1$, or, which is the same, $V\gamma$, where V is the difference of potential between the terminals of the motor. Accordingly, if A be the useful activity, we have for the efficiency of the motor the ratio $A/V\gamma$. We may call this the working efficiency of the motor.

Measurement of Working Efficiency of Motor.

To determine this ratio in any particular case the motor is run at the required speed, V is measured with a potential galvanometer, and γ with a current galvanometer, and their product taken, or $V\gamma$ is determined with some form of electrical activity-meter, while A is determined by means of a suitable ergometer. A very convenient and accurate friction ergometer may be formed by passing a cord once completely round the pulley of the motor, and hanging a weight on the downward end, while the other is made to pull on a spiral spring fixed at its upper end and provided with an index to show its extension. The weight is adjusted so that the motor runs at the required speed, while wasting all its work in overcoming the friction of the cord, and the extension of the spring is noted, and the corresponding pull found in the same units of force as those used in estimating the downward pull due to the weight. Let the weight

used in any experiment be taken in grammes, and be denoted by w, and let w' be the number of grammes required to stretch the spring by gravity to the same amount, then the total force overcome is in dynes $(w-w')g$, where g is the acceleration, in centimetres per second per second, produced by gravity at the place of experiment (at London $g = 981{\cdot}17$ nearly). If n be the number of revolutions per second, and c the circumference in cms. of the pulley at the part touched by the rope, the velocity with which this force is overcome is nc, and therefore the activity in ergs per second is $nc(w - w')g$. If A is reckoned in watts, we have the equation,

$$A = \frac{1}{10^7} nc(w - w')g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If $w - w'$ be taken in pounds, and c in feet, and n be the number of revolutions per *minute*, the activity in horse-power is given by

$$A = \frac{1}{33000} nc(w - w') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and in watts approximately by

$$A = {\cdot}0226nc(w - w') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9')$$

Generator Charging Storage Battery. Electrical Efficiency of Arrangement.

We have now considered cases in which electrical energy is transformed into mechanical work by means of motors working by electromagnetic action, and have seen that the whole electrical activity $E\gamma$ in the circuit is equal to the useful activity of the motor together with the unavailable part spent in heating the conductors in circuit, and in overcoming the frictional resistances opposing the motion of the motor. Part of the electrical energy developed by a generator may however be spent in effecting chemical decompositions in electrolytic cells placed in the circuit, as, for example, in charging a secondary battery or "accumulator." Each cell in which electrolytic action takes place, so

that the result is chemical separation at the plates of the constituents of the solution acted on, opposes a counter electromotive force to that causing the current, and the work done per second in each cell over and above that spent in heat according to Joule's law (p. 75), is equal to the product of this counter electromotive force into the strength of the current. In most cases the counter electromotive force exceeds the electromotive force required to effect the chemical decompositions, and the energy corresponding to the difference of electromotive force appears in the form of what has been called *local heat* in the electrolytic cells.

In the case of a secondary battery charged by the current from an electrical generator, which is the only case we shall here consider, the activity spent in the battery while it is being charged is equal to the product of the difference of potential existing between the terminals of the battery while the current is flowing, multiplied by the strength of the current. Let V be this difference of potential in volts, and γ the current strength in amperes, then $V\gamma$ joules is the whole work per unit of time spent in the battery. The whole activity spent in the circuit is $E\gamma$, or $V\gamma + \gamma^2 R$, where E is the total electromotive force of the generator, and R is the resistance of the generator and the wires connecting it with the secondary. Again, if E_1 volts be the electromotive force of the secondary battery, which may be measured by removing the charging battery for an instant and applying a potential galvanometer to the terminals of the secondary, the activity actually spent in charging the battery may be taken as $E_1\gamma$.

Hence the ratio of the activity spent in charging the battery to the whole activity in the circuit is $E_1/(V + R\gamma)$ or E_1/E, and the activity wasted in heating the conductors in circuit is $(E - E_1)\gamma$. This ratio E_1/E is the same as that found above in the case of a generator and a motor, and may be called as before the electrical efficiency of the arrangement.

Arrangement of Maximum Electrical Efficiency.

Hence, in order that as nearly as possible the whole electrical energy given out in the circuit may be spent in charging the battery, as many cells should be placed in circuit as suffice to nearly balance the electromotive force E of the generator, that is, the charging should be made to proceed as slowly as possible. In practice, however, a very slow rate of charging is not economical, as the work spent in driving the generator, if a dynamo- or magneto-electric machine, against frictional resistances would be greater than the useful work done in the circuit; and if the speed of the generator slackened for a little the battery would tend to discharge through it.

Effect of Increased E.M.F. in Circuit.

As in the case of the motor (p. 265), the electrical efficiency of the arrangement can be increased by increasing E and E_1, so that $E - E_1$ is maintained constant. E may, in the present case, be increased by running the generator faster, or by using a machine adapted to give higher potentials. As before, if E be increased to nE, while E_1 is changed to E' so that $nE - E' = E - E_1$, the electrical efficiency becomes $(n - 1)/n + E_1/nE$ as in (7) above.

Measurement of Energy Spent in Charging.

The electromotive force of a Faure or storage cell is rather over 2 volts when fully charged, but is considerably less when nearly discharged. When the cell is placed in the charging circuit, the counter electromotive force which it opposes rises quickly to a little less than this value, and thereafter gradually increases, while the charging current falls in strength. In order to measure, therefore, the whole energy spent in charging a secondary battery, we must either use some form of integrating energy-meter which gives accurate results, or measure, at short intervals of time, V with a potential galvanometer, and γ with a current galvanometer placed permanently in the circuit. After the battery has been charged, the total number of joules spent is obtained by multiplying each value of $V\gamma$ by the number of seconds between the instant at which the corresponding readings were taken and that at which the next pair of readings were taken, and adding all the results. Or, more exactly, values V and γ are obtained for each interval by finding the arithmetic means of the values of V and of γ at the beginning and end of each interval, and taking the product of these two means as the value of the activity for that interval. Each product is multiplied by the number of seconds in the corresponding interval, and the sum of the products is the whole energy spent. The integral work in joules having been thus estimated, the efficiency of the battery may be obtained by finding in the same manner the total number of joules given out in the external working circuit while the battery is discharging. The ratio of the useful work thus obtained to the whole work spent in charging is the efficiency of the battery. In discharging in an electric light circuit, the greatest economy is obtained when the resistance of the working part of the circuit is very great in comparison with that of the battery and main conductors. Neglecting the latter part of the resistance, we see that, if a large number of lamps are arranged in multiple arc, a large number of cells should also be joined in multiple arc, so that, while the requisite difference of potential is obtained, the resistance of the battery is still small in comparison with that of the external circuit.

Measurement of Activity Spent in Constant Current Circuits.

As regards the measurement of energy spent in electric light circuits, in which continuous currents are flowing, we have already sufficiently indicated above how this may be done. To find the activity, or work spent per unit of time, in any part of a circuit, we have only to

find the difference of potential, V, in volts between its extremities with a potential galvanometer, and the current, γ, in amperes flowing through it with a current instrument. If the activity be constant, we have simply to multiply $V\gamma$ by the number of seconds in any interval of time, to find the number of joules spent in that time in the part of the circuit in question. If the activity is variable, the whole energy spent in any time may be estimated by finding $V\gamma$ at short intervals of time, and calculating the integral as just explained.

Electrical Activity in Alternating Current Circuits.

So far we have been considering only measurements made in the circuits of batteries or of continuous current generators. Alternating machines in which the direction of the current is reversed two or three hundred times a second are, however, frequently employed, especially in electric light circuits, and it is necessary therefore to consider the methods of electrical measurement available in such cases.

The only electromagnetic instruments which can be used in alternating circuits are such as depend on the mutual force between two current-carrying conductors. Electrodynamometers generally, and current weighers, such as those described in Chap. VII, are instruments which act on this principle, and can be used both in alternating and in continuous current circuits. We have only to indicate here how they can be applied to measure currents, differences of potential, and activity in constant or alternating current circuits.

Practical Instruments

In practical work the instruments on this principle usually employed are such as require to have their con-

stants determined by comparison with standard instruments, such as a standard tangent galvanometer, or a standard dynamometer, and are dealt with in Chap. VII. We may here mention, in addition, Siemens' electrodynamometer, in which a suspended coil is acted on by a fixed coil, and the strength of the current deduced by means of a table of values for different angles, from the torsion which must be given to a spiral spring to bring the coil back to the zero position.

When an instrument on this principle is arranged for use as an activity-meter, one set of coils, the fixed or the movable, is made of thick wire so as to carry the whole current in the circuit, while the other set is made of high resistance and is connected to the two ends of the part of the circuit in which the electrical activity is to be measured. In this case the force or couple required to restore the movable coils to the zero position is proportional to the product $V\gamma$ of the difference of potential and current, that is to the activity, for that part of the circuit; and if the instrument has been properly graduated this can be at once read off in watts, or in any other chosen units of activity. Instruments of this kind have been made by Professors Ayrton and Perry, Sir William Siemens, and Lord Kelvin. Lord Kelvin's form * is a modification of the balance described above (Chap. VII.), in which the main current is sent through one set of the mutually acting coils, Activity-Meters.

* For a full description see the author's Smaller Treatise, Chap. VII.

which are therefore of low resistance; while the other set of coils are of high resistance and are applied at the terminals of the portion of the circuit in which the activity is to be measured, and therefore carry a current proportional to the difference of potential between those two points.

Measurements in Alternate Current Circuits.

We shall now consider the measurement of currents and differences of potential, and therefore also of electrical energy in the circuits of alternating machines or of transformers. In all such circuits the march of the

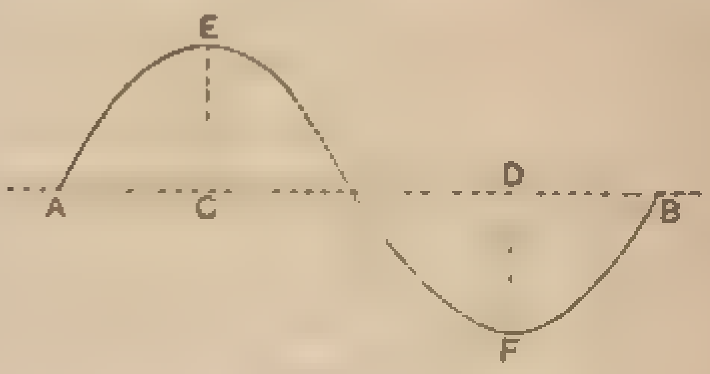

FIG. 147.

current in each complete alternation may be stated roughly as a rise from zero to maximum in one direction, then a diminution to zero, then a change of sign and a rise to maximum in the opposite direction, followed by a diminution again to zero. The law according to which these changes take place is more or less complex in the various cases, and the complete mathematical representation of the current strength at any time would require an application of Fourier's method of representing any arbitrary periodic function,

by means of an infinite series of simple harmonic terms of the form $A_i \sin (int - e_i)$, where n is 2π divided by the period T of a complete alternation, A_i and e_i constants and i any integer. It has been found experimentally by M. Joubert that the variation of electromotive force in a Siemens' alternating machine can be expressed by the single harmonic term $E \sin nt$, where we reckon t from the instant at which the electromotive force was zero when changing from the direction reckoned as negative to that reckoned as positive. The values of $E \sin nt$ are shown graphically by the ordinates of the curve in Fig. 147, t being measured from A along AB. The maximum and minimum ordinates CE, DF are in length numerically equal to the electromotive force E. This law applies fairly to many alternating machines, and we assume its truth in most of what follows (see however, Hopkinson's experiments on rapid cycles of magnetization, curves A, Chap XIII. Sect. III.). The more general case can be dealt with, when results for it are interesting or useful. By means of a proper contact arrangement, which makes connection with an electrometer at different instants during an alternation, the values of the difference of potential between the terminals at these instants can be obtained. If the difference of potential does not follow the simple law of signs, the simple harmonic constituents can be deduced by the method of *Least Squares* (see Appendix), from a sufficient number of such observations.

Law of E.M.F. in Alternate Current Circuits.

The current strength is affected by the action of self-induction to a greater or less extent in all such

machines independently of the disposition of the external circuit, especially if the revolving armature contains iron; but, as shown below, it follows, with a difference in phrase, the same law as does the electromotive force. The effect of variations in the field magnets produced by the rotating armature has also in a rigorous theory to be taken into account, but this effect, in well-designed machines without iron in their armatures is not great, and where experiments have been made to detect it, has been found to be slight, and we shall therefore neglect it.

Mean Current in Alternating Circuit.

Writing then γ for the current, at a time t, reckoned from the instant at which the current was zero, we have

$$\gamma = A \sin nt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The whole quantity of electricity generated in a half period $T/2$ is therefore

$$\int_0^{T/2} \gamma dt = A \int_0^{T/2} \sin nt dt = \frac{AT}{\pi} \quad . \quad . \quad . \quad (12)$$

Hence if C denote the mean current in that time, we have

$$\gamma_m = \frac{2A}{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Measurement of Mean Current.

Now if an electrodynamometer be placed in the circuit so that the same current passes through both its fixed and movable coils, the current in both will be reversed at the same instant, and their mutual action will be the same for the same current strength, and will be proportional to C^2 that is to $A^2 \sin^2 nt$. If the period of the alternation be small in comparison with the period of free oscillation of the movable coil system of the dynamometer, the mutual action of the fixed and movable coils

will be the same as if a continuous current γ' given by the equation

$$\gamma'^2 = \frac{1}{T}\int_0^T \gamma^2 dt = \frac{A^2}{T}\int_0^T \sin^2 nt dt. \quad . \quad . \quad . \quad (14)$$

Ratio of True Mean Current to Square Root of Mean Square of Current.

were kept flowing through them. But by integration

$$\gamma'^2 = \frac{A^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and substituting from (13) in this equation, we get

$$\gamma_m = \frac{2\sqrt{2}}{\pi}\gamma' = \cdot 9003\,\gamma'. \quad . \quad . \quad . \quad . \quad (16)$$

In order therefore to find the actual mean current strength in the circuit of an alternating machine from the value of γ' given by a current dynamometer we must multiply the latter by ·9; in other words the mean current strength is 9/10 of the strength of the continuous current which would give the same deflection. The product, if γ' has been taken in amperes, multiplied by the number of seconds in any interval of time during which the machine has been working uniformly on the same circuit, will give the number of coulombs of electricity which has flowed through the circuit in that time.

Measurement of Difference of Potential by Idiostatic Electrometer.

The measurement of differences of potential is however attended with more difficulty on account of the effect of the self-induction of any electromagnetic instrument which can be applied to the circuit for this purpose. The following method of employing a quadrant electrometer for this purpose has been used

Idiostatic Use of Electrometer.

by M. Joubert.* The needle of the instrument is left uncharged, and the charging rod connected with it and used as a third electrode. If the needle be connected to a point in the circuit at which the potential is V relatively to the outside case, one pair of quadrants at a point at which the potential is V_1, and the other pair at a third point where the potential is V_2, and if D be the deflection of the spot of light corresponding to the angle (supposed small), through which the needle has been turned against the bifilar suspension then subject to the caution below we have (Vol. I. p. 297)

$$D = k(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right) \quad . \quad . \quad (17)$$

where k is a constant. M. Joubert connected the needle (and case) to the pair of quadrants at potential V_1, so that

$$D = \frac{k}{2}(V_1 - V_2)^2 . \quad . \quad . \quad . \quad . \quad (18)$$

This equation is applicable, whatever the law of the electrometer, provided k be determined by a process of calibration with known differences of potential.

It has been found by Professors Ayrton and Perry and Mr. Sumpner † that when a quadrant electrometer is used idiostatically the metallic cheeks left where the guard-tube is cut away for the needle exert an influence on the needle in its unsymmetrical position when

* *Comptes Rendus*, July 1880. *Annales de Chimie et de Physique*, May, 1883.

† *Phil. Trans. R.S.*, A. 1891.

deflected which renders the formula (17) seriously inaccurate. It may be used however without correction for values of V up to about 100 volts. In quadrant electrometers now (1892) being manufactured, the guard-tube is dispensed with.*

A multicellular or vertical voltmeter † may (preferably) be used instead of the quadrant electrometer, except when three points at different potentials are to be connected to the electrometer at the same time. Any doubt as to the applicability of the expression on the right of (18), with k a constant, is avoided, for in these instruments the values of different deflections on the scale have been fixed by experiment.

If the terminals of the electrometer employed be connected to any two points in the circuit of a machine in which the period of alternation is short in comparison with the free period of the needle, the couple acting on the needle will be at each instant proportional to the second power of the difference $V_1 - V_2$ of potential existing between these two points at that instant. Also as in the similar case of the dynamometer above, the deflection of the needle will be the same as that which would be produced by a constant difference of potential V' given by the equation

$$V'^2 = \frac{1}{T}\int_0^T (V_1 - V_2)^2 dt \quad . \quad . \quad . \quad . \quad (19)$$

* To Messrs. Ayrton and Perry's instrument (18) was applicable only when the distance apart of the quadrants was 3·9 mms. The distance, if any, of the quadrants apart for which the formula is correct, should be found by experiment for each electrometer used in this manner.

† See Smaller Treatise Chap. VII., and Vol. I., p. 301.

Difference of Potential True Mean, and Square Root of Mean Square.

If we denote the actual mean difference of potential by V_m, then since the difference of potential follows the same law of variation as the current we get also

$$V_m = \cdot 9003 V' \quad . \quad . \quad . \quad . \quad . \quad (20)$$

If we know the resistance in the part of the external circuit between the points at which the electrometer electrodes are applied, then calling this resistance R, and supposing that this part of the circuit contains no motor or other arrangement giving a back electromotive force, and that the ratio of its self-induction to the period of alternation is zero or negligible in comparison with R, we have for the mean value of the current V_m/R, and thus by means of an electrometer alone we can measure not only the difference of potential between the ends of, but also the current in, that portion of the circuit.

Enhanced Resistance to Alternating Current distinguished from Impedance.

It is to be noticed that the resistance of the conductors in circuit is less the greater the frequency of alternation. This variation, as explained at p. 325 above, is due to the fact that as the alternation increases in rapidity the current is more and more confined by inductive action to the outer strata of the conductor, which is therefore virtually reduced in section. This is not to be confounded with the fictitious increase of resistance seen in the expression $\sqrt{R^2 + \bar{n}^2 L^2}$ (see p. 667 below) which arises directly from the electromotive force of self-induction; but is a real increase of the value of R for the current in question. (See table of the resistances of conductors at different periods of alternation in Appendix.)

Mean Activity in Alternating Circuit.

Denoting by A_m the mean value of the electrical activity in this part of the circuit, still supposing the self-induction of this part to be negligible, we have plainly

$$A_m = \frac{1}{RT}\int_0^T (V_1 - V_2)^2 dt = \frac{V'^2}{R} \quad . \quad . \quad . \quad . \quad (21)$$

In the same way, since the value of the electrical activity at any instant is $\gamma^2 R$, we have from the results of experiments made by an electrodynamometer,

$$A_m = \frac{R}{T}\int_0^T \gamma^2 dt = \gamma'^2 R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

From these two results we get

$$A_m = V'\gamma' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

That is

The true mean value of the electrical activity is equal to the product of the square root of the mean square of the difference of potential, by the square root of the mean square of the current strength. It can therefore be determined by means of an electrometer and an electrodynamometer of negligible self-induction without its being necessary to know the resistance.

Effect of Self-Induction.

We shall now consider the case in which the self-induction cannot be neglected. Let R be the total resistance in the circuit, γ the current flowing in it at the time t, E the total electromotive force of the machine, and L the inductance for the whole circuit, that is, the number which multiplied into $d\gamma/dt$ gives the electromotive force opposing the increase or diminution of the current. We shall suppose L a constant, although there can be no doubt that in some alternating machines its value is different in different positions of the armature. The iron cores of the field magnets act to a greater or less extent as cores for the armature coils, and as the magnetic susceptibility of iron is a function of the strength of the magnetizing current, L, which is the magnetic induction through the armature produced per unit of its own current, must vary accordingly.

Theory of Circuit Containing Simple Harmonic E.M.F.

Still for certain alternators which have no iron in their armatures the variation of L with the position of the armature is slight.* It will also be assumed that there are no masses of metal in which local currents can be generated moving in the field. On these assumptions the equation of the current is

$$R\gamma = E - L\frac{d\gamma}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

But by the law which we have assumed for the machine,

$$E = n\eta \sin nt = E_0 \sin nt \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where η is a constant such that E_0 is the maximum value of E for the given speed. Substituting in (24) we get

$$L\frac{d\gamma}{dt} + R\gamma = E_0 \sin nt \quad . \quad . \quad . \quad . \quad . \quad (26)$$

which integrated becomes

$$\gamma = A\epsilon^{-\frac{R}{L}t} + \frac{E_0}{\sqrt{R^2 + n^2L^2}} \sin(nt - e) \quad . \quad . \quad (27)$$

where

$$\sin e = \frac{nL}{\sqrt{R^2 + n^2L^2}}, \cos e = \frac{R}{\sqrt{R^2 + n^2L^2}} \quad . \quad . \quad (28)$$

The term $A\epsilon^{-\frac{R}{L}t}$ is only important immediately after the circuit is closed, and will therefore be neglected.

We may remark that if L were equal to zero (27) would reduce to $\gamma = E_0/R \cdot \sin nt$, which corresponds to (12) above.

Mean Current.

From (27) we get for the mean current

$$\gamma_m = \frac{2E_0}{T(R^2 + n^2L^2)^{\frac{1}{2}}}\int_{e/n}^{e/n+T/2} \sin(nt - e)\,dt = \frac{2E_0}{\pi(R^2 + n^2L^2)^{\frac{1}{2}}} \cdot (29)$$

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* See the discussion on Mr. W. M. Mordey's paper on "Alternating Current Working" Inst. of Elect. Eng. May, 1889 (*The Electrician*, May 24, 31, June 7, Aug. 2, 1889).

Mean Square of Current Strength.

Also for the mean square of the current strength as given directly by an electrodynamometer we have by (27) the equation

$$\gamma'^2 = \frac{E_0^2}{T(R^2 + n^2L^2)}\int_0^T \sin^2(nt - e)dt = \frac{1}{2}\frac{E_0^2}{R^2 + n^2L^2} \quad . \quad (30)$$

and we have therefore as before, the relation

$$\gamma_m = \cdot 9003\gamma'.$$

Difference of Phase of Current and E.M.F.

From (27) we see that the effect of self-induction is to diminish every value of the current in the ratio of $E_0/(R^2 + n^2L^2)^{\frac{1}{2}}$ to E_0/R, and to produce a retardation of phase which measured in time is e/n seconds; that is, the resistance is virtually increased in the ratio $(R^2 + n^2L^2)^{\frac{1}{2}}/R$, and the current in following the law of sines passes through any value e/n seconds after it would have passed through the corresponding value if there had been no self-induction. If in Fig. 147 above the ordinates of the curve of sines represent the values of the current at different instants of time, when L is zero, the current would be represented for any given value of L by diminishing the ordinates of the curve all in the proportion of R to $R^2 + n^2L^2$, and shifting the curve along AB from left to right through a distance equal to η. It is plain also that, for any finite resistance R, by diminishing T, that is, by increasing the speed of the machine, the current can, by (27), be made to approach the limiting value

$$\gamma = \frac{\eta}{L}\sin\left(nt - \frac{\pi}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Maximum Mean Current for Given Resistance.

which is independent of the resistance, and has a retardation of phase of $T/4$ seconds, a quarter period of a complete alternation. Hence integrating over a half period from zero current to zero current again, and dividing by $T/2$ we get for the maximum mean current

$$\gamma_m = \frac{2\eta}{\pi L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Mean Electrical Activity in Circuit.

To find the mean value A_m of the total electrical activity in the circuit, we have by (25), and (27)

$$A_m = \frac{1}{T}\int_0^T E\gamma dt = \frac{E_0^2}{(R^2 + n^2L^2)^{\frac{1}{2}}}\int_0^T \sin(nt - e)\sin nt\,dt$$

$$= \frac{1}{2}\frac{E_0^2R}{R^2 + n^2L^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Hence by (30) if the activity is spent in heat

$$A_m = \gamma^2 R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

that is, the true mean value of the total electrical activity is equal to the mean square of the current strength multiplied by the total resistance in circuit. This also applies to part of a circuit.

Maximum Activity. 1. For Given Speed and Inductance. 2. For Given Resistance.

It may easily be shown, from (33), by the ordinary method that the total activity in the circuit is greatest when $R=nL$, that is, *for a given speed and a given value of* L, the activity is a maximum when $R=nL$. It must be observed however that *for a given resistance* R the activity is greater the smaller the value of T, that is, the greater the speed. When R has the value nL we have, by (27) $e=\pi/4$; that is, the retardation of phase is then one-eighth of the whole period.*

Supposing the electromotive force and current, though periodic, not to follow the simple sine law, then as we have seen above we may represent either by a Fourier series. Thus

Complex-Periodic E.M.F.

$$E = \Sigma E_i \sin (int - e_i) \quad . \quad . \quad . \quad . \quad . \quad (25')$$

where i is an integer, and takes all the values required for the simple components which make up the periodic function which E is of the time.

For equation of current we have now instead of (26)

$$L\frac{d\gamma}{dt} + R\gamma = \Sigma E_i \sin (int - e_i) \quad . \quad . \quad . \quad . \quad (26')$$

Law of Current.

of which the solution is

$$\gamma = A\epsilon^{-\frac{R}{L}t} + \sum \frac{E_i}{\sqrt{R^2 + i^2n^2L^2}} \sin (int - e_i - \phi_i) \quad . \quad (27')$$

where

$$\sin \phi_i = \frac{inL}{\sqrt{R^2 + i^2n^2L^2}}, \quad \cos \phi_i = \frac{R}{\sqrt{R^2 + i^2n^2L^2}} \quad . \quad (28')$$

* The conclusions as to maximum work and retardation of phase, as well as most of the theoretical results stated above as to the action of alternating machines, were first we believe given by M. Joubert, *Comptes Rendus*, 1880. The problems of alternating machines joined in series, or in parallel, or as motors, were considered by Dr J Hopkinson in a lecture to the Inst. of Civil Engs. 1883, and in a paper "On the Theory of Alternating Currents," Soc Tel Engs., and Els., Nov 1884. The principal results of this latter paper are reproduced below. See also Mr. Mordey's paper *loc. cit.* p. 666 above.

As before we may neglect the exponential term in the solution.

If now we wish to find the mean square of the current strength we have only to square the series on the right of (27′) and integrate over the whole compound period, $2\pi/n$, that is, over an interval which is the least common multiple of the periods of the components. Now it can be easily shown that an integral of the form

$$\int \sin(int - e_i - \phi_i) \sin(jnt - e_j - \phi_j)dt$$

vanishes when taken over an interval $2\pi/n$, unless $i = j$. For the product under the integral sign can by elementary trigonometry be resolved into the difference of two cosines, each yielding a simple integral, which obviously vanishes.

Then since

Mean Square of Current = Sum of Mean Squares of Components.

$$\frac{1}{2\pi/n} \frac{E_i^2}{R^2 + i^2n^2L^2} \int_0^{2\pi/n} \sin^2(int - e_i - \phi_i)dt = \frac{1}{2} \frac{E_i^2}{R^2 + i^2n^2L^2}$$

we get for the mean square of the current

$$\gamma'^2 = \frac{1}{2} \sum \frac{E_i^2}{R^2 + i^2n^2L^2} \quad . \quad . \quad . \quad . \quad . \quad (30')$$

that is, the mean square of the current is the sum of the mean squares of the currents which would be given by the components of γ if each existed separately.

The mean activity is given by the equation

Mean Activity = Sum of Mean Activities of Components.

$$A_m = \frac{1}{2\pi/n} \int_0^{2\pi/n} \sum E_i \sin(int - e_i)$$

$$\cdot \sum \frac{E_i}{\sqrt{R^2 + i^2n^2L^2}} \sin(int - e_i - \phi_i)dt.$$

If the multiplication of the two series on the right is performed a number of integrals of the form

$$\frac{1}{2\pi/n} \int_0^{2\pi/n} \sin(int - e_i) \sin(jnt - e_j - \phi_j)dt$$

are obtained, all of which vanish as before except those for which $i = j$. But we have

$$\frac{1}{2\pi/n}\frac{E_i^2}{\sqrt{R^2 + i^2n^2L^2}}\int_0^{2\pi/n}\sin(int - e_i)\sin(int - e_i - \phi_i)dt$$

$$= \tfrac{1}{2}\frac{E_i^2\cos\phi_i}{\sqrt{R^2 + i^2n^2L^2}} = \tfrac{1}{2}\frac{E_i^2R}{R^2 + i^2n^2L^2}$$

by (28′). Hence

$$A_m = \frac{R}{2}\sum\frac{E_i^2}{R^2 + i^2n^2L^2} \quad . \quad . \quad . \quad . \quad . \quad (33')$$

that is, the mean activity is the sum of the mean activities which the component currents would give separately.

Also by (30′) and (33′)

$$A_m = \gamma'^2R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34')$$

Practical Application of Result. The practical importance of this result lies in this, that it proves that any method of measuring power which is demonstrated for a current following the simple sine law of variation with the time, is also true for any periodic current whatever, inasmuch as such a current can be regarded as made up of simple sine currents of different periods. For example, the generality of the method, given below, of measuring power in the circuit of a transformer can be inferred from this result, as has been remarked by Prof. Perry.*

Circuit with two E M F s of same Period. If in the circuit there be two sources of electromotive force of the same period T, but of different phases; for example, two machines driven so as to have the same period of alternation, the solution here given applies. For the two electromotive forces combine to give a single electromotive force of the same period as the components, but differing in phase from either; so that, to use the solution it is only necessary to take this resultant electromotive force as $E_0 \sin nt$, reckoning the time from an instant at which $\sin nt$ is zero and increasing. If the difference of phases be 2ϕ reckoned in angle, the interval between the successive instants at which a component is increasing through zero is $2\phi/n$. Hence taking the zero of reckoning of time midway between these two instants we may denote the two compo-

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* *Phil. Mag.*, Aug. 1891.

nents by $E_1 \sin(nt+\phi)$, $E_2 \sin(nt-\phi)$. Calling their resultant $E_0 \sin(nt-\psi)$, we have

$$E_0 \sin(nt-\psi) = E_1 \sin(nt+\phi) + E_2 \sin(nt-\phi) \quad (35)$$

By elementary trigonometry we get

$$\left.\begin{aligned} E_0^2 &= E_1^2 + E_2^2 + 2E_1E_2\cos 2\phi \\ \text{and} \quad \tan\psi &= \frac{E_2 - E_1}{E_1 + E_2}\tan\phi \end{aligned}\right\} \quad . \ . \ . \ (36)$$

When $\phi = o$, $\psi = o$, and $E_0 = E_1 + E_2$, as is evident without calculation, since the machines are then in the same phase. If $E_1 = E_2$, that is if the machines are equal, the resultant is in phase halfway between its components. When this is the case we have also

$$E_0 = 2E_1 \cos\phi \quad . \ . \ . \ . \ . \ . \ . \ (37)$$

which when $\phi = o$ gives, as it ought, $E_0 = 2E_1$.

Considering still two unequal machines, and remembering that when the value of the resultant electromotive force is increasing through zero, the value of the current is given by (27), that then the electromotive force of the leading machine is $E_1 \sin(nt+\phi+\psi)$, and that of the following machine $E_2 \sin(nt-\phi+\psi)$, we have for the mean activity A_{1m} of the leading machine Two Unequal E.M.F.s of Different Phase.

$$A_{1m} = \frac{1}{T}\int_0^T E\gamma\, dt$$

$$= \frac{E_0E_1}{T(R^2+n^2L^2)^{\frac{1}{2}}}\int_0^T \sin(nt-\epsilon)\sin(nt+\phi+\psi)\,dt$$

$$= \frac{1}{2}\frac{E_0E_1}{(R^2+n^2L^2)^{\frac{1}{2}}}\cos(\phi+\psi+\epsilon)$$

$$= \frac{1}{2}\frac{E_0E_1}{R^2+n^2L^2}\{R\cos(\phi+\psi) - nL\sin(\phi+\psi)\} \quad . \ (38)$$

To find the mean activity of the following machine we have only to change the sign of ϕ in this expression. We get

$$A_{2m} = \frac{1}{2}\frac{E_0E_2}{R^2+n^2L^2}\{R\cos(\phi-\psi) + nL\sin(\phi-\psi)\} \quad (39)$$

If the machines be equal $E_1 = E_2$, and $\psi = 0$, so that

$$A_{1m} = \frac{E_1^2 \cos\phi}{R^2 + n^2L^2}(R\cos\phi - nL\sin\phi) \quad . \quad . \quad (40)$$

$$A_{2m} = \frac{E_1^2 \cos\phi}{R^2 + n^2L^2}(R\cos\phi + nL\sin\phi). \quad . \quad . \quad (41)$$

Tendency of Two Equal Machines to Opposition of Phase.

Since ϕ is less than $\pi/2$, both $\cos\phi$ and $\sin\phi$ are positive, and therefore the following machine does more work than the leading machine. Hence, unless each is completely controlled by the prime-mover, the leading machine will increase its lead, and this will go on until $2\phi = \pi$, when the two machines will be in exactly opposite phases, and will exactly neutralise one another. This tendency to assume opposition of phase depends on the difference $A_{2m} - A_{1m}$, and this having the factor $nL/(R^2 + n^2L^2)$, has a maximum value, for a given resistance and a given period of alternation, when $nL = R$.

The machines thus arrange themselves so that no current passes in the wires joining their terminals, and these wires alternate in relative potential with the period of the machines, and each is at any instant very approximately at one potential throughout. It might therefore be inferred that if a working circuit be joined from one wire to the other, a current will pass through that circuit, and that the two machines will control one another so as to keep in the same phase in supplying it. We shall consider this case as a further example of the theory.

Two Equal Alternators in Parallel.

Let 2ϕ be the difference of phase with reference to the external circuit, so that at time t, $E\sin(nt+\phi)$, $E\sin(nt-\phi)$ are the electromotive forces of the two machines, γ_1, γ_2 the currents, L the coefficient (supposed constant) of self-induction for each, r the resistance of each machine from one point of attachment to the other point, and R the resistance of the external circuit. We shall suppose that the external circuit has no sensible self-induction, and that the whole work there developed is spent in overcoming resistance, for example, in lighting glow-lamps. By considering the circuit through each machine and the external resistance,* remembering that the current in the latter is $\gamma_1 + \gamma_2$, and therefore the difference of

* According to Kirchhoff's rule, p. 88 above, taking into account the electromotive force of self-induction in each circuit.

potential between the terminals $R(\gamma_1 + \gamma_2)$, we find, as at p. 489 above, the equations

$$\left.\begin{aligned} L\frac{d\gamma_1}{dt} + r\gamma_1 + R(\gamma_1 + \gamma_2) &= E\sin(nt + \phi) \\ L\frac{d\gamma_2}{dt} + r\gamma_2 + R(\gamma_1 + \gamma_2) &= E\sin(nt - \phi) \end{aligned}\right\} \quad . \quad (42)$$

Adding and subtracting we get

$$\left.\begin{aligned} &L\frac{d}{dt}(\gamma_1 + \gamma_2) + (2R + r)(\gamma_1 + \gamma_2) = 2E\cos\phi \,.\, \sin nt \\ &L\frac{d}{dt}(\gamma_1 - \gamma_2) + r(\gamma_1 - \gamma_2) = 2E\sin\phi \,.\, \cos nt \end{aligned}\right\} \quad (43)$$

Solving these we find as in (27)

$$\gamma_1 + \gamma_2 = \frac{2E\sin\phi}{\{(2R + r)^2 + n^2L^2\}^{\frac{1}{2}}}\sin(nt - \epsilon) \quad . \quad . \quad (44)$$

$$\gamma_1 - \gamma_2 = \frac{2E\sin\phi}{\{r^2 + n^2L^2\}^{\frac{1}{2}}}\cos(nt - \epsilon') \quad . \quad . \quad . \quad (45)$$

where

$$\tan\epsilon = \frac{nL}{2R + r}, \quad \tan\epsilon' = \frac{nL}{r} \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Hence if A_{1m} be the activity of the leading machine

$$A_{1m} = \frac{E}{2T}\int_0^T (\gamma_1 + \gamma_2 + \gamma_1 - \gamma_2)\sin(nt + \phi)dt$$

$$= \frac{E^2}{T}\left\{\frac{\cos\phi}{\{(2R + r)^2 + n^2L^2\}^{\frac{1}{2}}}\int_0^T \sin(nt - \epsilon)\sin(nt + \phi)dt\right.$$

$$\left. + \frac{\sin\phi}{(r^2 + n^2L^2)^{\frac{1}{2}}}\int_0^T \cos(nt - \epsilon')\sin(nt + \phi)dt\right\}$$

$$= \tfrac{1}{2}\frac{E^2}{(2R + r)^2 + n^2L^2}\{(2R + r)\cos^2\phi - nL\sin\phi\cos\phi\}$$

$$+ \tfrac{1}{2}\frac{E^2}{r^2 + n^2L^2}(r\sin^2\phi + nL\sin\phi\cos\phi) \quad . \quad . \quad (47)$$

Activity of each Machine.

The mean activity A_{2m} of the following machine may be got from A_{1m} by altering the sign of ϕ throughout the expression on the right. Hence

$$A_{2m} = \frac{1}{2}\frac{E^2}{(2R+r)^2+n^2L^2}\{(2R+r)\cos^2\phi + nL\sin\phi\cos\phi\}$$

$$+\frac{1}{2}\frac{E^2}{r^2+n^2L^2}(r\sin^2\phi - nL\sin\phi\cos\phi) \quad . \quad . \quad (48)$$

Theory of Synchronizing of Two Parallel Alternators.

$A_{1m}-A_{2m}$ is positive, that is more work is done by the leading than by the following machine. The lead will therefore tend to zero, and the machines to settle down into coincidence of phase with reference to the external circuit, that is, into opposite phases with reference to their own circuit, which agrees with the result already obtained.

Alternating Motor.

We shall consider only one more case of this theory, that of an alternating motor connected by its terminals to two conductors upon which an alternating difference of potential is impressed by other machines. Let the motor be started so as to have the same period of alternation. Then denoting by R the resistance of the motor-armature and the leads, up to the point at which the difference of potential is impressed, by L the self-inductance for the same part of the circuit, by $E_1\sin(nt+\phi)$, the impressed difference of potential at time t, by $E_2\sin(nt-\phi)$, the back electromotive force of the motor at the same instant, we have for the equation of the current

$$L\frac{d\gamma}{dt} + R\gamma = E_1\sin(nt+\phi) - E_2\sin(nt-\phi) \quad . \quad (49)$$

Theory of Alternating Motor.

This equation differs only in the sign of E_2 from that from which (38) and (39) above are deduced. Hence taking the value of A_2m in (41) we have for the mean electric activity received by the motor

$$A_{2m} = \frac{1}{2}\frac{E_0E_2}{R^2+n^2L^2}\{R\cos(\phi-\psi) + nL\sin(\phi-\psi)\} \quad (50)$$

where

$$\left.\begin{aligned} E_0 &= (E_1^2+E_2^2-2E_1E_2\cos 2\phi)^{\frac{1}{2}} \\ \tan\psi &= -\frac{E_1+E_2}{E_1-E_2}\tan\phi \end{aligned}\right\} \quad . \quad . \quad . \quad (51)$$

The second of (51) gives

$$\cos\psi = (E_1 - E_2)\cos\phi/E_0, \quad \sin\psi = -(E_1 + E_2)\sin\phi/E_0,$$

and these values substituted in (50) yield

$$A_{2m} = \frac{1}{2}\frac{E_2}{R^2 + n^2L^2}\{E_1(R\cos 2\phi + nL\sin 2\phi) - E_2R\} \quad (52)$$

Maximum Value of Activity of Motor.

Now 2ϕ being the difference of phase cannot be numerically greater than π, and therefore the work *received* by the motor is less when 2ϕ is negative than when it is positive, that is, less when the motor is leading than when it is following. Hence the motor will tend to run slower when leading and faster when following, or the difference of phase will tend towards zero. Also so long as 2ϕ is not far from zero A_{2m} is less the greater the lead, and greater the greater the lag, and in nearly the same proportion. Hence when the machines are once in phase any small deviation is opposed by a proportional corrective tendency. This depends almost entirely on the term involving the factor $nL/(R^2 + n^2L^2)$ in the value of A_{2m} given in (52), and therefore for a given resistance R, and period of alternation T, has its greatest value when $nL = R$, or $L/R = T/2\pi$.

Writing in (52)

$$\sin 2\phi' = R/(R^2 + n^2L^2)^{\frac{1}{2}}, \quad \cos 2\phi' = nL/(R^2 + n^2L^2)^{\frac{1}{2}} \quad (53)$$

we get

$$A_{2m} = \frac{1}{2}\frac{E_2}{R^2 + n^2L^2}\{E_1(R^2 + n^2L^2)^{\frac{1}{2}}\sin 2(\phi + \phi') - E_2R\} \quad (54)$$

Explanation of Self-Synchronizing Action.

which is obviously a maximum when $\phi + \phi' = \pi/4$. We have then

$$A_{2m} = \frac{1}{2}\frac{E_2}{R^2 + n^2L^2}\{E_1(R^2 + n^2L^2)^{\frac{1}{2}} - E_2R\} \quad . \quad . \quad (55)$$

The value of A_{2m} is positive if

$$\frac{E_1}{E_2} > \frac{R}{(R^2 + n^2L^2)^{\frac{1}{2}}}$$

which may be the case even if $E_2 > E_1$. Hence we have the curious result that an alternating machine may act as a motor even if its electromotive force be greater than the impressed or driving electromotive force.

A qualitative explanation of the results given above for two alternators can be given graphically by taking the areas of curves drawn to represent the activity at each instant. From these it will at once appear which machine is doing the greater amount of work. The reader may easily construct these curves by drawing for each machine, from the curves giving the current and electromotive force at each instant a new curve, the ordinates of which are the products of the corresponding ordinates of the former.

Comparison of Theory and Experiment. The theory just given of the working of alternating machines in the same circuit is (apart from notation and mode of statement) substantially that due to Dr. J. Hopkinson.* Its conclusions were verified by him in 1884, in experiments made with two De Merten's machines made for the lighthouse at Tino. Some very striking experiments are described by Mr Mordey† in a paper on alternate current working, which contains moreover much interesting practical information on this subject. Some difference of opinion has been expressed as to whether Mr. Mordey's results are in accordance with the mathematical theory. It is to be remembered however that the theory does not take into account the action of the armature currents in the field-magnets, nor of the variation of self-induction. The subject requires further investigation.

We may apply, as we have already done repeatedly above (see for example p. 186), the mode of treatment adopted for the whole circuit to a part of it, taking for E the impressed electromotive force on the part of the circuit considered, and for R and L the proper values for that part only. We find that the effect of self-induction is virtually to increase the resistance from R to $\sqrt{R^2 + n^2L^2}$, that is to substitute impedance for resistance, and to produce a difference of phase between the current and the impressed electromotive force given also by (27) and (28). But the resistance of a con-

* *Proc. Soc. Tel. Eng. and El.* Nov. 1884, also see Thompson's *Dynamo Electric Machinery*, p. 691 *et seq.*

† *Inst. El. Eng.* May 1889 (*The Electrician*, May 24, 31, June 7, Aug. 2, 1889).

ductor is the activity spent in it by unit current in producing heat; hence by (22) and (34) the resistance in this sense is not increased.

Measurement of Current in Alternating Circuit.

The impedance of a current electrodynamometer or current balance, through both coil systems of which flows the whole current in the main circuit, cannot, if it be low (as it generally is) in comparison with that of the rest of the circuit, affect appreciably the strength of the current by its introduction; and since the whole current passes through both sets of coils, the instrument will give the mean square of the current passing.

Measurement of Difference of Potential in Alternating Circuit.

It may be otherwise however with a fine wire instrument used as a shunt to measure the difference of potential between two points of the circuit. The inductance of such an instrument may be considerable, and if it be used alone its impedance will seriously affect the result. Since the value of the impedance depends on the period of alternation, it will have different values when connected to circuits in which the periods are different. To obviate the uncertainty and inconvenience arising from this cause, the instrument is made sensitive enough to allow a considerable non-inductive resistance to be joined in series with its own coils. This makes the value of $R/\sqrt{R^2+n^2L^2}$ approximately unity. Some calculations made by Prof. T. Gray, for Lord Kelvin's vertical scale voltmeter, give for this ratio with only the resistance of the instrument (640 ohms) included, and a period of alternation of $\frac{1}{100}$ of a second, the value ·9976, which is within $\frac{1}{4}$ per cent. of unity. Plainly the error caused by the impedance in this case is small with any period commonly employed, and can be made still

smaller by the introduction of non-inductive resistance. The difference of phase between the currents through the coils of the instrument, and the difference of potential [given by (28) above] is therefore small. This difference of phase, it is to be remembered, does not affect the value of the mean square of the difference of potential, provided the amplitude be corrected for the effect of inductance.

Measurement of Activity in Alternating Circuit.

It is however of importance in the action of a wattmeter, of which one coil is placed in the main circuit, and the other as a shunt between the extremities of the portion of the circuit in which the activity is to be estimated. For let the circuit divide into two parts, each forming a derived current with the other, and L_1, L_2, R_1, R_2, ${}_m\gamma_1$, ${}_m\gamma_2$, be the inductances, the resistances and the maximum currents in the two parts, ${}_m\gamma$ the maximum total current in the circuit, and ϵ'_1, ϵ'_2 the difference of phase between ${}_m\gamma$ and ${}_m\gamma_1$, ${}_m\gamma_2$, respectively, then the general formula (91), p. 187, above for the difference of phase θ between the total current in the circuit and the applied electromotive force at the common terminals of a multiple arc gives in this case

$$\tan\theta = \frac{n\{L_1(R_2^2+n^2L_2^2)+L_2(R_1^2+n^2L_1^2)\}}{R_1(R_2^2+n^2L_2^2)+R_2(R_1^2+n^2L_1^2)} \quad . \quad . \quad (56)$$

and by (89), p. 186, the difference of phase ϵ_1 between the impressed electromotive force and the current γ_1 is given by

$$\tan\epsilon_1 = \frac{nL_1}{R_1}$$

Hence for the lag in phase ϵ'_1 $(=\epsilon_1-\theta)$ of the current γ_1 behind the main current we have

$$\tan\epsilon'_1 = \tan(\epsilon_1-\theta) = \frac{n(L_1R_2-L_2R_1)}{R_2(R_1+R_2)+n^2L_2(L_1+L_2)} \quad . \quad (57)$$

An interchange of suffixes in this result of course gives $\tan\epsilon'_2$.

A method of determining the difference of phase between the currents in two branches of the same circuit, or between two currents of the same period, will presently be explained.

By (90), p. 186, the value of the square of the maximum total current is easily found to be

Difference of Phase between Currents in Double Arc.

$$_m\gamma^2 = E^2 \frac{(R_1+R_2)^2 + n^2(L_1+L_2)^2}{(R_1^2+n^2L_1^2)(R_2^2+n^2L_2^2)} \quad . \quad . \quad . \quad (58)$$

and by (89)

$$_m\gamma_1^2 = \frac{E^2}{R_1^2+n^2L_1^2}, \quad _m\gamma_2^2 = \frac{E^2}{R_2^2+n^2L_2^2} \quad . \quad . \quad . \quad (59)$$

so that

$$\frac{_m\gamma_1^2}{R_2^2+n^2L^2} = \frac{_m\gamma_2^2}{R_1^2+n^2L_1^2} = \frac{_m\gamma^2}{(R_1+R_2)^2+n^2(L_1+L_2)^2} \quad (60)$$

Condition that Difference of Phase may be Insensible

Hence if either L_1, L_2, be both small or $L_1/L_2 = R_1/R_2$, the difference of phase between the two currents $_m\gamma_1$, $_m\gamma_2$, will be insensible. If the first condition is fulfilled both parts of the circuit will have currents agreeing in phase with the difference of potential between the terminals, and on the usually allowable supposition of negligible mutual inductance, a wattmeter whose coils are included in them will measure accurately the power expended. It will, on the same supposition, also measure accurately the power expended *while the wattmeter is on circuit*, if the ratio $R/\sqrt{R^2+n^2L^2}$ be approximately unity for the fine wire circuit, since the main current passes through the other coil, and it can be shown that the deflection will be the same as would be produced by a constant activity A_m given by the equation

$$A_m = \frac{1}{T}\int_0^T V\gamma dt \quad . \quad . \quad . \quad . \quad (61)$$

where V, γ, are the values of the difference of potential and the current at time t. If also $\sqrt{R^2+n^2L^2}$ for the

thick wire coil be small in comparison with the same quantity for the part of the main circuit in which the activity is being measured, the inclusion of the wattmeter will not affect the circuit, and the activity shown by the instrument may be taken as that existing when it is not applied.

Apparent and True Mean Activity.

The general problem of finding the ratio of the apparent to the true mean activity as shown by the wattmeter can now be solved with great ease. For let A, B, be the points at which the terminals of the fine wire coil system are attached to the main circuit; let R_1, R_2, L_1, L_2, be the resistances and inductances of the fine wire and thick wire circuits between A, B, and γ_1, γ_2, the currents in them; then by (27) if the difference of potentials between the terminals AB is $E_0 \sin nt$,

$$\left.\begin{aligned}\gamma_1 &= \frac{E_0}{(R_1^2 + n^2L_1^2)^{\frac{1}{2}}} \sin (nt - e_1) \\ \gamma_2 &= \frac{E_0}{(R_2^2 + n^2L_2^2)^{\frac{1}{2}}} \sin (nt - e_2)\end{aligned}\right\} \quad . \quad . \quad . \quad (62)$$

with $\tan e_1 = nL_1/R_1$, $\tan e_2 = nL_2/R_2$.

The current through the fine wire coil is therefore the same as if the resistance in its circuit between the points A, B, were without inductance, and the difference of potential had the value obtained by multiplying the above value of γ_1 by R_1. Hence if A'_m be the apparent activity

Apparent Mean Activity as shown by Wattmeter.

$$A'_m = \frac{1}{T}\frac{E_0^2 R_1}{(R_1^2 + n^2L_1^2)^{\frac{1}{2}}(R_2^2 + n^2L_2^2)^{\frac{1}{2}}}\int_0^T \sin(nt - e_1)\sin(nt - e_2)dt$$

$$= \frac{1}{2}\frac{E_0^2 R_1 \cos (e_1 - e_2)}{(R_1^2 + n^2L_1^2)^{\frac{1}{2}}(R_2^2 + n^2L_2^2)^{\frac{1}{2}}} \quad . \quad . \quad . \quad (63)$$

that is the apparent activity is $\frac{1}{2}$ the product of the maximum values of the two currents by the resistance R_1 of the fine wire branch, and by the cosine of the phase-angle between the currents.

The true mean activity A_m would be obtained if the current through the fine wire branch had the value $E_0 \sin nt/R_1$. In that case the phase-angle between the two currents would be e_2. Hence as before

$$A_m = \tfrac{1}{2}\frac{E_0^2 \cos e_2}{(R_2^2+n^2L_2^2)^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

True Mean Activity.

Hence

$$\frac{A_m}{A'_m} = \frac{(R_1^2+n^2L_1^2)^{\frac{1}{2}}}{R_1} \cdot \frac{\cos e_2}{\cos(e_1-e_2)} \quad . \quad . \quad . \quad . \quad (65)$$

Ratio of Apparent to True Mean Activity.

Since $\sin e_1 = nL_1/(R_1^2+n^2L_1^2)^{\frac{1}{2}}$, $\cos e_1 = R_1/(R_1^2+n^2L_1^2)^{\frac{1}{2}}$, we may calculate $\cos e_2/\cos(e_1-e_2)$ in terms of L_1, L_2, R_1, R_2. Thus we obtain

$$\frac{A_m}{A'_m} = \frac{R_2}{R_1} \cdot \frac{R_1^2+n^2L_1^2}{R_1R_2+n^2L_1L_2} = \frac{1+n^2\tau_1^2}{1+n^2\tau_1\tau_2} \quad . \quad . \quad . \quad (65')$$

where τ_1, τ_2, are written for L_1/R_1, L_2/R_2, respectively the so called time constants of the two parts of the circuit.

Now in general $\tau_1 < \tau_2$, hence as a rule the wattmeter will give a too high result.*

Mr. Blakesley † has shown how the angle ϕ of difference of phase between the currents in two such branches may be measured. We have seen that a current dynamometer in any branch measures the mean square of the current in that branch. This has the value ${}_m\gamma^2/2$, where ${}_m\gamma$ denotes the maximum value of the current in the branch. Now let ${}_m\gamma_1$, ${}_m\gamma_2$ be the maximum currents in the two branches, and let two dynamometers be arranged one to measure ${}_m\gamma_1^2/2$, and the other ${}_m\gamma_2^2/2$, and let a third be placed, with one coil in one, and the other coil in the other of the two branches in question. The action on the third dynamometer at any instant will be proportional to $\gamma_1\gamma_2 \cos\phi$. Hence the instrument will give a reading proportional to $\frac{1}{2}\,{}_m\gamma_1\,{}_m\gamma_2 \cos\phi$. If then D_1, D_2, D_3, be the readings of the dynamometers, A, B, C, their constants, so that

Measurement of Difference of Phase between Two Alternating Currents.

$$D_1/A = {}_m\gamma_1^2/2, \quad D_2/B = {}_m\gamma_2^2/2, \quad D_3/C = \tfrac{1}{2}\,{}_m\gamma_1\,{}_m\gamma_2 \cos\phi,$$

we get

$$\cos\phi = \frac{D_3}{\sqrt{D_1D_2}} \cdot \frac{\sqrt{AB}}{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

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* This result was first stated (without proof) by Prof. Ayrton *Proc. Soc. Tel. Eng. and El.* Feb. 1888.

† *Electrician*, Oct. 2, 1885, and *Phil. Mag.* April 1888.

If the dynamometers are direct reading so that $A = B = C = 1$, the factor $\sqrt{ABC}$ is unity.

Of course if three dynamometers are not available a single dynamometer may be used to take the three readings in succession (or to eliminate error several sets of readings may be taken and combined). In that case $A = B = C$ and

$$\cos\phi = \frac{D_3}{\sqrt{D_1 D_2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66')$$

Mr. Blakesley* has also given an exceedingly simple and ingenious method of measuring the total activity spent in the primary circuit of a transformer, that is of finding the whole electrical work done per unit of time in feeding the secondary, and directly or indirectly in dissipation.

Transformer.

A transformer consists, as is well known, of a primary and secondary circuit wound round a core of laminated iron, in general in such a manner that as nearly as possible all lines of magnetic induction, which pass through any spire of one of the coils, also pass through every other spire of the same or the other coil. It is not however safe to assume that this is always the case, and, as has been pointed out by Prof. Perry, serious errors may arise through making the assumption in all circumstances.

Blakesley's Method of Measuring Activity in a Transformer.

Now let a current dynamometer be placed in the primary circuit, and another be arranged with one coil in the primary and the other coil in the secondary circuit. Then if D_1 be the deflection-reading of the first instrument, A the constant of reduction of the readings to (current)^2, D_{12} and B the corresponding quantities for the other instrument (both deflections being taken positive), N_1, N_2, the number of turns in the primary and secondary respectively, R_1, R_2, their resistances, and A_m the mean activity to be measured

$$A_m = R_1 \frac{D_1}{A} + R_2 \frac{N_1}{N_2} \frac{D_{12}}{B} \quad . \quad . \quad . \quad . \quad . \quad (67)$$

under certain assumptions.

This method is applicable whatever may be the law of variation of current.

Theory of Method.

The equations of a primary and secondary circuit given in (77), p. 183, may be used, and may be modified by writing E for

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* *Phil. Mag.* 1891 or *Proc. Phys. Soc.* 11, pt. 2, 1891.

$E_0 \sin nt$, since we make here no assumption as to the mode of variation of the current or electromotive force. These equations hold for any primary or secondary whether or not containing iron. We shall first also write B_1, B_2, for the total inductions through a single turn of the primary and the secondary respectively, and N_1, N_2, for the number of turns in the coils. Thus we can write the equation referred to in the form

$$\left.\begin{aligned} R_1\gamma_1 + N_1 \frac{d\mathsf{B}_1}{dt} = E \\ R_2\gamma_2 + N_2 \frac{d\mathsf{B}_2}{dt} = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Proof of Formula on Assumption of Equal Inductions.

Then we have

$$R_1 \frac{D_1}{A} = \frac{R_1}{T}\int_0^T \gamma_1^2 dt = \frac{1}{T}\int_0^T E\gamma_1 dt - \frac{N_1}{T}\int_0^T \gamma_1 \frac{d\mathsf{B}_1}{dt}\,dt$$

or

$$\frac{1}{T}\int_0^T E\gamma_1 dt = R_1 \frac{D_1}{A} + \frac{N_1}{T}\int_0^T \gamma_1 \frac{d\mathsf{B}_1}{dt}\,dt \quad . \quad . \quad , \quad (69)$$

by the first of (68).

If D_2 be, in the same way, the reading of the second instrument, taking account of the sign of the deflection, and B its constant,

$$R_2 \frac{D_{12}}{B} = \frac{R_2}{T}\int_0^T \gamma_1\gamma_2 dt = -\frac{N_2}{T}\int_0^T \gamma_1 \frac{d\mathsf{B}_2}{dt}\,dt \quad . \quad . \quad (70)$$

by the second of (68).

If now we assume that $\mathsf{B}_1 = \mathsf{B}_2$, we get from the last equation

$$\frac{N_1}{T}\int_0^T \gamma_1 \frac{d\mathsf{B}_1}{dt}\,dt = -R_2 \frac{N_1}{N_2}\frac{D_2}{B}.$$

Substituting from this in (69) we find

$$\frac{1}{T}\int_0^T E\gamma_1 dt = R_1 \frac{D_1}{A} - R_2 \frac{N_1}{N_2}\frac{D_{12}}{B} \quad . \quad . \quad . \quad . \quad (71)$$

and the quantity on the left is the mean value of the total activity. Thus the total activity is given by the expression on the right in terms of the reading of the dynamometers.

Signs of Dynamometer Readings.

It is to be noticed that since γ_1, γ_2, are on the whole in opposite directions, the sign of D_2 must be opposite to that of D_1. Thus the second term of the expression on the right of (71) is really negative, and the total rate of working is greater than the first term, which represents the activity spent in heat in the circuit. Hence if we agree to take the positive numerical value of the reading of the second instrument for D_{12}, we may, putting A_{1m} for the mean activity on the primary, write (70) in the form

$$A_{1m} = R_1 \frac{D_1}{A} + R_2 \frac{N_1}{N_2} \frac{D_{12}}{B} \quad . \quad . \quad . \quad . \quad (71')$$

This method and result were given by Mr. T. H. Blakesley for a transformer on the assumption that the currents followed the simple sine law of variation: in the demonstration given above no assumption at all is made except that $\mathsf{B}_1 = \mathsf{B}_2$. The method is therefore so far applicable to any transformer whatever the law of variation followed by the current provided B_1 may be taken as equal to B_2. This was first pointed out by Prof. Ayrton and Mr. J. F. Taylor*, whose method of proof is similar to that here given.

Proof on Assumption of Constant Permeability.

It has been shown also by Prof. Perry † that this method holds even if B_1 be not equal to B_2, provided we suppose the permeability constant during a cycle. In this case the equations of current may be written in the form,

$$\left.\begin{aligned} R_1\gamma_1 + L_1 \frac{d\gamma_1}{dt} + M \frac{d\gamma_2}{dt} &= E \\ R_2\gamma_2 + L_2 \frac{d\gamma_2}{dt} + M \frac{d\gamma_1}{dt} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (72)$$

since L_1, L_2, M, do not vary in a cycle if the permeability does not. Hence multiplying the first of these by γ_1, and calculating the mean value of each quantity by integrating over a whole period, we get

$$A_{1m} = \frac{1}{T}\int_0^T E\gamma_1 dt = \frac{R_1}{T}\int_0^T \gamma_1^2 dt + \frac{M}{T}\int_0^T \gamma_1 \frac{d\gamma_2}{dt} dt. \quad (73)$$

since the integral of $\gamma_1 d\gamma_1/dt \, . \, dt$ over a period is zero.

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* *Proc. Phys Soc.* Dec. 1891.

† *Phil. Mag.* Aug. 1891.

But if we multiply the second of (72) by γ_1 and take mean values as before, we find

$$\frac{R_2}{T}\int_0^T \gamma_1\gamma_2 dt + \frac{L_2}{T}\int_0^T \gamma_1 \frac{d\gamma_2}{dt} dt = 0,$$

since the last integral vanishes as before. Thus

$$\frac{M}{T}\int_0^T \gamma_1 \frac{d\gamma_2}{dt} dt = -\frac{M}{L_2}\frac{R_2}{T}\int_0^T \gamma_1\gamma_2 dt \quad . \quad . \quad . \quad (74)$$

Substituting in (73) we get

$$A_{1m} = \frac{R_1}{T}\int_0^T \gamma_1^2 dt - \frac{M}{L_2}\frac{R_2}{T}\int_0^T \gamma_1\gamma_2 dt \quad . \quad . \quad (75)$$

or putting in the readings of the dynamometers (taking D_{12} positive as before)

$$A_{1m} = R_1\frac{D_1}{A} + R_2\frac{M}{L_2}\frac{D_{12}}{B} = R_1\frac{D_1}{A} + R_2\frac{N_1}{N_2}\frac{D_{12}}{B} \quad . \quad (76)$$

since approximately $M = N_1N_2$, $L_2 = N_2^2$. This is the same result as before, but obtained under a different assumption, not however involving any hypothesis as to the mode of variation of the current.

It is to be observed that this supposition of no variation of L_1, L_2, or M, is equivalent to supposing that all the activity is employed in generating heat in the two circuits. For if the second of (72) be multiplied by γ_2, and then integrated for mean values, it gives *Constant Permeability involves Zero Dissipation in Iron Core.*

$$\frac{R_2}{T}\int_0^T \gamma_2^2 dt + \frac{M}{T}\int_0^T \gamma_2 \frac{d\gamma_1}{dt} dt = 0.$$

This added to (73) gives

$$A_{1m} = \frac{R_1}{T}\int_0^T \gamma_1^2 dt + \frac{R_2}{T}\int_0^T \gamma_2^2 dt + \frac{M}{T}\int_0^T \frac{d(\gamma_1\gamma_2)}{dt} dt,$$

and the last term vanishes since the integration is round a closed cycle. Thus

$$A_{1m} = \frac{R_1}{T}\int_0^T \gamma_1^2 dt + \frac{R_2}{T}\int_0^T \gamma_2^2 dt. \quad . \quad . \quad . \quad (75)$$

or the total mean activity is equal to the rate of generation of heat in the secondary plus that in the secondary. The activity could in this case be equally well measured by placing a current dynamometer in the primary, and another in the secondary, as by Mr. Blakesley's method.

Hysteresis

The supposition thus made by Prof. Perry therefore excludes all dissipation of energy otherwise than by direct heating of the circuits by the currents. It has been urged by him that on the analogy of the behaviour of ordinary bodies under strain produced by stress varied in rapid cycles, there ought to be no dissipation of energy due to lagging of the magnetization behind the magnetic force in the cycle, as explained at p. 212 above or, as it is now called, *hysteresis* action, in iron subjected to rapid cycles of magnetic stress. On this view magnetic like elastic hysteresis is only important in slow cycles. This analogy appears a reasonable one, but any opinion founded on it must be tested by direct experiment. Now it has been given as the result of experiment by several observers* that there is in rapid cycles dissipation of energy in the core of the same order of magnitude as in slow cycles; but that there is much less when the transformer is loaded by closing the secondary circuit through a low resistance, than when the secondary circuit is open.

Hysteresis in Rapid Cycles.

This result is questioned by Prof. Ewing, who gives as the result of experiments on a transformer core, an anchor ring made of iron wire insulated to prevent eddy currents, that, for the same frequency of reversal and limits of magnetization, the loss by magnetic hysteresis is just as great when the transformer is heavily loaded, as when its secondary circuit is open.†

The rate of loss by hysteresis is however in all cases small in comparison with the whole activity. [See also Chap. XIII.]

Assuming the truth of Mr. Blakesley's formula as deduced from the hypothesis of no magnetic leakage, we can find the amount of energy spent in eddy currents and magnetic hysteresis in the iron.

Assuming for simplicity that the dynamometers are direct reading instruments, or if not that D_1, D_{12}, are reduced readings expressing each a mean square of current measured in amperes, so that the constants A, $B=1$, then R_1, R_2, being taken in ohms,

* Warburg and Honig. *Wied Ann* 20, 1883.

† Tanakadate, *Phil. Mag* Sept. 1889. See also Ewing, *Magnetism in Iron and other Metals*, §§ 83, 180. See also below, Chap. XIII., Section III.

Energy Spent in Hysteresis.

A_{1m} will be given in watts. If now we suppose a third dynamometer placed in the secondary circuit, and D_2 in like manner be its reading, we shall have

$$R_2 D_2 = \frac{R_2}{T}\int_0^T \gamma_2^2 dt.$$

Thus we have

$$A_{1m} = R_1 D_1 + R_2 D_2 + R_2\left(\frac{N_1}{N_2} D_{12} - D_2\right) \quad . \quad . \quad (77)$$

The two first terms on the right express the whole work done in heating the wires of the primary and secondary, the third term that spent in heating the iron by eddy currents and hysteresis.

If R'_2 be the resistance of the external part of the secondary, and the work done in that be wholly spent in heat, the energy there spent is $R'_2 D_2$. Thus if e be the electrical efficiency of the transformer

$$e = \frac{R'_2 D_2}{R_1 D_1 + R_2 \frac{N_1}{N_2} D_{12}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

Difference of Potential between Terminals of Primary.

From the expression for A_{1m} can be found at once the difference of potential between the terminals of the primary. For if R'_1 be the external resistance of the primary circuit between its terminals, we have instead of (68)

$$\left.\begin{aligned} R'_1\gamma_1 + N_1\frac{d\mathbf{B}_1}{dt} &= V \\ R_2\gamma_2 + N_2\frac{d\mathbf{B}_2}{dt} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (68')$$

Squaring the first of these we get

$$V^2 = R'^2_1\gamma_1^2 + N_1^2\left(\frac{d\mathbf{B}_1}{dt}\right)^2 + 2R'_1 N_1 \gamma_1 \frac{d\mathbf{B}_1}{dt}.$$

Hence if $\overline{V^2}$ be the mean square of the difference of potential V,

$$\overline{V^2} = \frac{R'^2_1}{T}\int_0^T \gamma_1^2 dt + \frac{N_1^2}{T}\int_0^T\left(\frac{d\mathbf{B}_1}{dt}\right)^2 dt + \frac{2R'_1 N_1}{T}\int_0^T \gamma_1 \frac{d\mathbf{B}_1}{dt}\,dt$$

The first integral is, as we have seen above, $K_1^2D_1$, and the third $2K_1R_2D_{12}/N_2$. The second integral can be found by the method of ..., since B_2 is taken as equal to B_1. Thus

$$\frac{1}{T}\int_0^T\left(\frac{d\mathsf{B}_2}{dt}\right)^2 dt = \frac{R_2^2}{N_2^2}\frac{1}{T}\int_0^T \gamma_2^2 dt = \frac{R_2^2}{N_2^2}D_2 .$$

Substituting these values for the integrals we get

$$V^2 = K_1^2D_1 + \frac{R_2^2}{N_2^2}D_2 + 2K_1R_2\frac{1}{N_2}D_{12} \quad . \quad . \quad (79)$$

The above results are all independent of the law of variation of the current and involve only the assumption $\mathsf{B}_1 = \mathsf{B}_2$. They are due to Mr Blakesley, but were first proved by methods similar to those used above, by Prof. Ayrton and Mr. Taylor in their paper above referred to.

Measurement of Activity by Current-meter only. In any practical case of measurement of power in which a wattmeter is inapplicable, if the actual resistance of the portion of the circuit considered is known and the mean square of the current can be measured with accuracy, the product of the two will as shown above (p. 668) be the true mean value of the activity if that is spent in heat. This of course will be given in watts, if the resistance is taken in ohms and the current in amperes.

As we have seen above, the proper mean value of the current, and of the difference of potential, and therefore also of the activity, can be found for any part of a circuit in the case of negligible self-induction, either by means of an electrodynamometer, or by means of an electrometer, when the resistance of one part of the circuit is known.

Method when Resistance is Unknown. When the resistance is unknown or uncertain, as for example in the case of incandescence lamps, the current and difference of potential may be measured for the lamp circuit in the following manner. A coil of german silver wire, having a resistance considerably greater than that of the lamps as arranged, constructed so as to have no self-inductance, is connected in series with a current-meter between the terminals of the machine so as to be a shunt on the lamps. The lamps are brought to their normal brilliancy, and the mean square γ'^2 of the current through the german silver wire measured. If R be the resistance of this wire, including, if appreciable, the resistances of the current-meter and its connections, and R be great in comparison with the self-inductance of the current-meter divided by T, we have for the mean square V'^2, of the difference of potential between the terminals

of the lamp system, the value γ'^2R^2. The current-meter is now employed to measure the whole current flowing to the lamps while their brilliancy is kept the same. Denoting the mean square of this current by γ_1^2, we have for the value A_m of the mean activity spent in the lamp system

$$A_m = V^1\gamma' = \gamma'\gamma_1'R \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

Testing Dynamo-Machines. Method of Messrs. Hopkinson

Messrs. J. and E. Hopkinson* have employed the following method of testing the efficiency of dynamo-machines, which obviates the difficulty of measuring accurately the mechanical power transmitted to the driving shaft of a dynamo by a steam engine or other motor. Two equal dynamos of the type to be tested are used, and one of these is run as a motor at the required speed and with the proper amount of electrical activity in the circuit. This can be adjusted by suitably varying the magnet resistances of one of the machines. The motor is made to spend the available activity which it gives out in driving the generator, and the difference in power required is supplied by a steam- or other engine, and measured by a Hefner-Alteneck dynamometer, or by any other similar method by which the difference of tensions of the two parts of the belt is determined. This latter amount of power represents the losses in transmission, and added to the power returned to the generator by the motor gives the mechanical power required to drive the generator. The errors inherent in the determination of mechanical power transmitted to a driven shaft are thus made to affect only the comparatively small balance of power

* *Phil. Trans. R.S.* Part I. 1886.

and the efficiency is obtained to a much higher percentage of accuracy.

The whole electrical power $E\gamma$ developed by the generator is then found by calculating that spent on each part of the circuit from the observed differences of potential between the terminals of the generator and motor, the current in the circuit, and the known resistances of the different parts of the machines. By adding to this the power w, in watts, wasted in the machine, the power spent in driving it is obtained and hence at once the gross efficiency $E\gamma/(E\gamma + w)$.

Then the sum of the powers developed in the armature and magnets of each machine, and in the leads and other resistances in the circuit, subtracted from the power transmitted from the engine and measured by the dynamometer, gives a balance which represents the total loss in the circuit over and above those here enumerated. This is made up of power wasted in the iron cores of the armatures and in the pole pieces in consequence of hysteresis or eddy currents, in reversals of the currents in the sections of the armatures, in connexions, in sparking if any, and in the friction of the bearings and brushes. Half of this balance may be taken as spent in each machine. The whole power spent in driving the generator is therefore the sum of the whole electrical power $E\gamma$ given out in the circuit, and half the balance, w say. Thus the efficiency is

Gross Efficiency of Dynamo

$$e = \frac{E\gamma}{E\gamma + w} = 1 - \frac{w}{E\gamma}, \text{ nearly.}$$

Swinburne's Method.

Mr Swinburne measures electrically the loss of power w here described, and requires only one machine of the type to be tested. The magnets of the machine are excited separately so that the armature is under the induction which would exist if the machine were working under the load specified for it. The machine is then driven by a small dynamo which furnishes current at the electromotive force of the machine just sufficient to drive it at the requisite speed, without any load beyond that involved in w, namely the losses in eddy currents,

hysteresis, and friction in the machine being tested. The speed can be adjusted as in the tests above described by suitably varying the resistance of the magnet circuit. The power spent on the machine by the small dynamo is determined electrically in the ordinary way by measuring the number of volts difference of potential between the terminals and the current in amperes. The former will of course be approximately the full electromotive force of the machine when working under the prescribed load. The power thus determined, diminished by that spent in heat in the armature (which is generally negligible), is the waste power w required.

Efficiencies

The efficiency can then be found by calculating the total electrical activity in the circuit when the machine is running under the prescribed load, by adding to the activity in the external circuit the electrical activities in the armature and magnets, found in watts by multiplying the resistance of each part in ohms by the square of the current in amperes. Call this electrical activity $E\gamma$, as above, p. 645. Then the mechanical power spent in driving is $E\gamma + w$. The gross efficiency of the machine is thus $E\gamma/(E\gamma + w)$. The electrical efficiency of the arrangement is $E_1\gamma/E\gamma$, if E_1 be the difference of potential between the terminals of the external circuit. Finally the net efficiency is $E_1\gamma/(E\gamma + w)$.

Sumpner's Method of Testing Transformers.

On the analogy of the Messrs. Hopkinson's method of testing dynamos just described, Dr. Sumpner has based the following method of testing power supplied to transformers. Two equal transformers have their primary coils c_1c_2 joined in parallel across the terminals of an alternating dynamo as shown in Fig. 148, and their secondaries C_1C_2 also joined in parallel between the points AB. Non-inductive resistances r and R are included in the primary and secondary circuits as shown.

Supposing the transformers to be alike, and the primary circuits to have the same resistance, the magnetizing currents will be the same in both, and there will be equal electromotive forces at any instant in the secondaries. Thus no current will flow in the secondary circuit whatever the resistance R. A non-inductive resistance r in the primary of either will cause the currents in the primaries to be different, and if r is in the circuit of C_2 a current will flow in the secondary which will load the transformer c_1C_1, and help to magnetize the core of c_2C_2, thus raising the electromotive force in the primary of that transformer.

If however the transformers be somewhat different, for example, so that (to take Dr. Sumpner's example) No. 1 converts from 100 to 2100 volts, and the other from 100 to 2000 volts,

then there will be an electromotive force of 100 volts in the circuit of the secondaries which will produce any desired current if R be properly adjusted.

Determination of Waste Power.

If then with two unequal transformers the current flowing through the secondaries be of the proper amount, each transformer will be fully loaded, but one, the more powerful, No. 1 say, will transform up, and the other down. That is the former will take energy from the mains, the other will return energy to the mains. The power-losses occurring in the double transformation are then, in the aggregate, the difference between the power taken by No. 1, and that given back by No. 2,

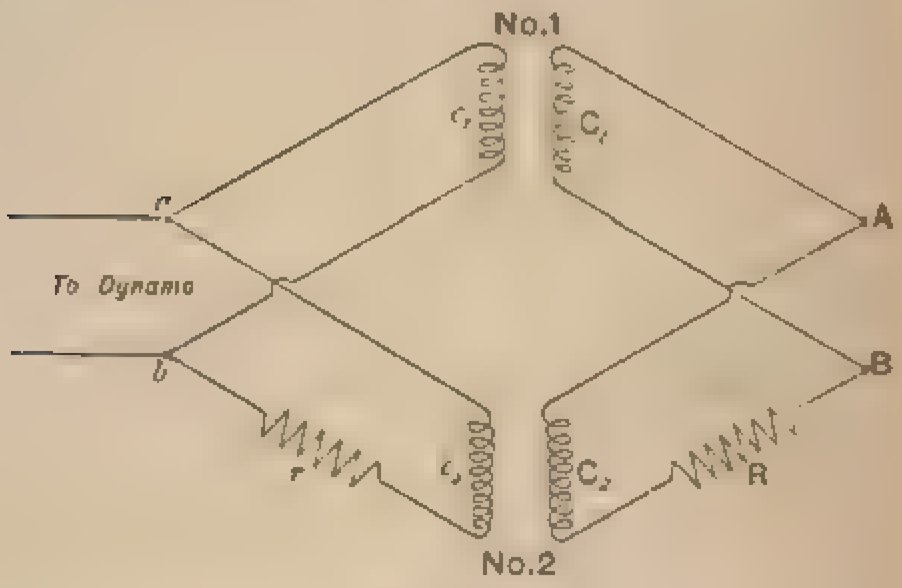

FIG. 148.

diminished by the amount absorbed in the resistance R, and by the amount spent in heating the connecting wires and instruments applied.

It is only necessary therefore, in order to obtain w, to measure the balance of power supplied to the system at ab, and correct it as described. This may be done with a wattmeter, the fine wire coil of which is placed across the terminals ab, and the current coil in one of the mains, or by the electrometer-method described below. To calculate the efficiency we have then only to find the power W, say, supplied to No. 1 transformer. This can be done nearly enough by measuring the load on either transformer, say by placing a wattmeter with its fine wire coils across ab, and its current coil in c, or by measuring the difference

of potential and current of any coil of either transformer. Then the efficiency of the double transformation, e^2 say, is given by

Calculation of Efficiency.

$$e^2 = \frac{W}{W + w} = 1 - \frac{w}{W},$$

nearly.

The efficiency of each transformer is approximately the square root of this, or

$$e = 1 - \frac{1}{2}\frac{w}{W} - \frac{1}{8}\frac{w^2}{W^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

nearly.

One great advantage of this method lies in the fact that a considerable error in the estimation of w can only slightly affect that of e^2 or e, if e be not very different from unity. This method as it stands is only applicable to two transformers the electromotive forces of the secondaries of which differ by at least twice the "drop" in difference of potential between the terminals of the secondary of either, when its load is raised from zero to the prescribed value. In the case of two similar transformers Dr. Sumpner uses a small additional transformer which is able to supply the waste w for the two large transformers to be tested. The primary of this is connected in series with an adjustable non-inductive resistance, x, across the main terminals ab, and the secondary is placed in, say, No. 2 transformer, in series with either c_2 or C_2, in place of the non-inductive resistance r or R

Case of Two Equal Transformers.

This small transformer will supply an amount of energy, depending on the value to which x is adjusted, sufficient to cause any required current to flow in the secondaries of the large transformers It is only necessary then to measure the energy given out by the small transformer by measuring the current and difference of potential on its primary and secondary, and further to measure as before the power supplied by the mains. The sum of these corrected as before will be w. Then W is measured as before for either of the large transformers and the efficiency is determined by (81) above.

Different arrangements will suggest themselves to the engineer carrying out these tests as suitable in the varying circumstances in which he may be placed by his instruments, &c.*

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* See a paper by Prof. Ayrton and Dr. Sumpner, *Electrician*, Oct. 7, 1892.

Three Voltmeter Method.

Prof. Ayrton and Dr. Sumpner* have given the following very elegant method of measuring the power given out in any portion of a circuit. It will be seen that it is intimately related to the electrometer method described below. Three points on the circuit are taken, two (Fig. 149) AB between which is the portion of the current in which the activity is to be found, while the portion BC consists of a non-inductive resistance of R ohms. Three alternate current volt meters of proper construction are used to give the mean squares of the differences between A and B, B and C, and A and C.

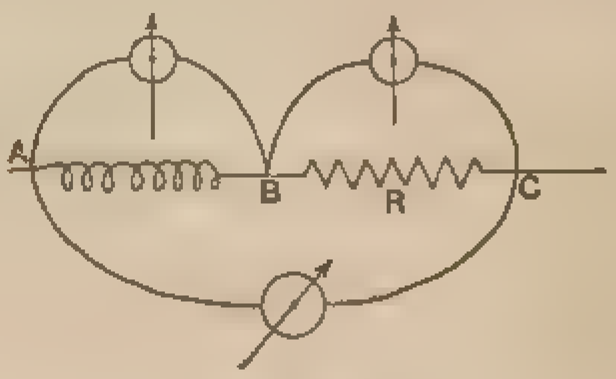


Fig. 149.

If D_1, D_2, D be the readings of these voltmeters each in volts, and A_m the mean activity in watts

$$A_m = \frac{1}{2R}(D^2 - D_1^2 - D_2^2) \quad . \quad . \quad . \quad (82)$$

For

$$A_m = \frac{1}{T}\int_0^T V_1 \gamma dt,$$

Theory of Method. if V_1 be the difference of potential between A and B at any instant. But if V_2 be the difference of potential existing at

* *Proc. R.S.* April 9, 1891 or *Electrician*, April 17, 1891.

the same instant between B and C we have $\gamma = V_2/R$. Hence

$$A_m = \frac{1}{RT}\int_0^T V_1 V_2 dt$$

The difference of potential between A and C is at the same instant $V_1 + V_2$, and we have

$$V_1 V_2 = \tfrac{1}{2}\{(V_1 + V_2)^2 - V_1^2 - V_2^2\}.$$

Hence

$$A_m = \frac{1}{2RT}\left\{\int_0^T (V_1 + V_2)^2 dt - \int_0^T V_1^2 dt - \int_0^T V_2^2 dt\right\},$$

or

$$A_m = \frac{1}{2R}(D^2 - D_1^2 - D_2^2) \quad . \quad . \quad . \quad . \quad . \quad (82)$$

It can be shown by the Theory of Errors of Observation that on the assumption of equal proportional errors in the quantities

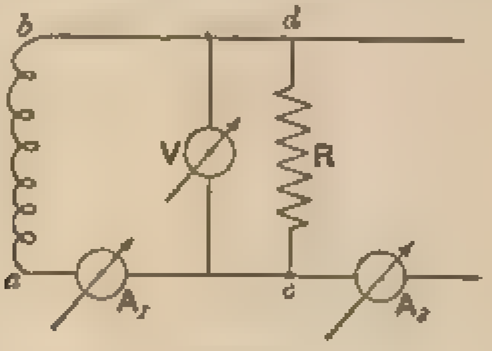


FIG. 150.

observed the best arrangement for this measurement is one in which the mean square of the difference of potential between A and B is equal to that between B and C. This however is an arrangement in which the power consumed in the non-inductive resistance is equal to the power measured.

Method with Two Current-meters and Voltmeter.

A modification of this method has been proposed by Prof. Ayrton and Dr. Sumpner in which two current meters A_1, A_2, and a voltmeter V are arranged as shown in Fig. 150. ab is the portion of the circuit in which the power is to be measured, cd a non-inductive resistance placed across its terminals, V is a voltmeter placed parallel to AB and CD, and measuring the mean square of the difference of potential between ac and bd.

If V be the difference of potential between a and b at any instant, and γ the current at that instant, the activity is

$$A_m = \frac{1}{T}\int_0^T V\gamma dt,$$

or if γ' be the current in cd at the same instant,

$$A_m = \frac{R}{T}\int_0^T \gamma\gamma' dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (83)$$

since $\gamma' = V/R$. But if D_1 be the reading of the current-meter A_1, D_2 that of A_2 (each giving the mean square of the current in amperes), we have

$$D_1 = \frac{1}{T}\int_0^T \gamma^2 dt,$$

and

$$D_2 = \frac{1}{T}\int_0^T (\gamma + \gamma')^2 dt = \frac{1}{T}\int_0^T (\gamma^2 + \gamma'^2 + 2\gamma\gamma')dt$$

$$= D_1 + \frac{D^2}{R^2} + \frac{2}{T}\int_0^T \gamma\gamma' dt,$$

if D be the reading of the voltmeter expressed in volts. Hence by (83)

$$A_m = \frac{R}{T}\int_0^T \gamma\gamma' dt = \frac{R}{2}\left(D_2 - D_1 + \frac{D^2}{R^2}\right) \quad . \quad . \qquad (84)$$

This was given* as an improvement upon a method proposed by Dr. J. A. Fleming in which a current-meter is placed in cd, and A_m is given by (84), with D_3 put for the reading of this current-meter, and used instead of the term D_2/R^2. The current-meter introduces a certain amount of inductance into cd, although this might be made negligible by taking cd large enough.

Electrometer Method Measurement Mean Square of Current

An electrometer may be used in the following manner to give the mean square of the current, and of the difference of potential for any part of a circuit, whether containing motors or arc lamps or any arrangement with or without counter-electromotive force or self-

* "Alternate Current and Potential Difference Analogies," *Phil. Mag.* Aug. 1891.

inductance. A coil of thick German silver wire (or to prevent sensible heating a set of two or more coils arranged in multiple arc) having no self-inductance is included in the part of the circuit considered, so that the current to be measured also flows through the wire. The mean square of the difference of potential between the ends of this resistance is measured as described above (p. 279) by connecting one pair of quadrants of the electrometer to one end, and the needle and the other pair of quadrants to the other end, and the mean square γ'^2 of the current by dividing by the square of the resistance of the wire. The mean square of the difference of potential between the terminals of the part of the circuit considered is then found in the same manner. A multicellular electrostatic voltmeter is very convenient for such measurements on account of its great range of sensibility (see Vol. I. Chap. V.).

and Mean Square of Difference of Potential.

The product is not generally to be taken as the mean square of the activity in the part of the circuit considered, for it is evident that in this case what is obtained is the value of

$$\frac{1}{T}\int_0^T V^2 dt \times \int_0^T \gamma^2 dt,$$

where V and γ are the difference of potential and the current at any instant. The square root of this quantity is not generally the same thing as

Square Root of Product of Mean Squares not True Mean Activity.

$$\frac{1}{T}\int_0^T V\gamma dt,$$

the true mean value of the activity. This is, however, given indirectly by the following method.*

Electrometer-Method of Determining Activity.

Let the two ends of the resistance coil of zero self-inductance and known resistance R be called A and B, and let the extremities of the portion of the circuit for which the measurements are to be made, be called C and D. One of the pairs of quadrants is connected to A, the other pair to B, and the needle to C, and the reading, d say, taken. The quadrants remaining as they were, the needle is connected to O, and the reading d' taken. Now if at any instant V_1 be the potential of A, V_2 of B, V'_1 of C, and V'_2 of D, we get if (17) above is applicable to the instrument (see p. 663 above)

$$\left.\begin{aligned} d &= \frac{k}{T}\int_0^T (V_1 - V_2)\left(V'_1 - \frac{V_1 + V_2}{2}\right) dt \\ d' &= \frac{k}{T}\int_0^T (V_1 - V_2)\left(V'_2 - \frac{V_1 + V_2}{2}\right) dt \end{aligned}\right\} \qquad (85)$$

and by subtraction and division by kR

$$\frac{d - d'}{kR} = \frac{1}{RT}\int_0^T (V_1 - V_2)(V'_1 - V'_2)dt \quad . \quad (86)$$

But it is clear that the expression on the right hand side of (86) is the true mean value of the activity required.

* This method is described by A. Potier, *Journal de Physique*, t. ix. p. 227, 1881, but was independently invented also by Prof. W. E. Ayrton, and Prof. G. F. Fitzgerald (see Prof Ayrton on "Testing the Power and Efficiency of Transformers," *Proc. Soc. Tel. Engs. and Els.*, Feb. 1888).

If $V'_1 - V_1$ be great in comparison with $V_1 - V_2$ and A, say, be connected with the case of the instrument, the first of becomes

$$d = \frac{k}{T}\int_0^T (V_1 \quad V_2)\, V'_1 dt \quad . \quad . \quad . \quad . \quad . \quad (87)$$

If A and D coincide $V'_1 + V_1$, and the activity in the part of the circuit between C and D is, by (61), given by (62) alone when put in the form

$$\frac{d}{kR} = \frac{1}{RT}\int_0^T (V_1 - V_2)\, V'_1 dt \quad . \quad . \quad . \quad . \quad (88)$$

This observation is due to Mr. Sayers, a pupil of Prof. Ayrton. It is thus possible in the case supposed to use an electrometer as a direct reading wattmeter.

If a quadrant electrometer is used as here explained, care must be taken to see that the equation (17) holds for the instrument (see p. 633 above). Dr. Hopkinson found (*Phil. Mag.* Ap. 1885) that the indications of his instrument were very exactly expressed by the equation

Precautions in Use of Quadrant Electrometer.

$$D = \frac{1}{1 + mV^2}(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right) \quad . \quad . \quad (89)$$

where m is a small constant. Hence for high values of V it is necessary to know and use this second constant if its value is sensible. The deviation from fulfilment of the ordinary equation here shown was found to be in great part due to downward electrical force on the needle caused by its hanging a little too low in the quadrants.

CHAPTER XIII

MAGNETIC MEASUREMENTS

SECTION I

MEASUREMENT OF INTENSE MAGNETIC FIELDS

Measurement of Field Intensity by Electromagnetic Force.

WE have seen (p. 118 above) that every element of a conductor carrying a current in a magnetic field is acted on by a force tending to move it in a direction at right angles to its length and to the direction of the magnetic induction at the element, and have stated how the magnitude of the force may be calculated in terms of the induction, the strength of the current and the position of the element. Hence, if we know the strength of the current flowing in a conductor placed in a magnetic field, and measure the force exerted in virtue of electromagnetic action on any element of the conductor, we can find the induction, that is, in air, the intensity of the field at the element. On this principle are founded the following simple methods, mainly suggested by Lord Kelvin, of determining in absolute measure the intensity of magnetic fields in dynamo machines or other electromagnetic apparatus.

We shall take first the case of two long straight pole faces oppositely magnetized and placed at a short distance apart facing one another, with their lengths vertical. In the middle

of the space between the poles a stout wire, w (Fig. 151), somewhat longer than the poles, so as to extend a little above and below them, is hung vertically by a cord four or five feet long, attached near its upper end to a fixed peg above, and is stretched by the weight, W, attached near its lower end. Two pendulums made of weights P_1, P_2, carried by fine threads, are hung from

Field between Two long Straight Pole Faces.

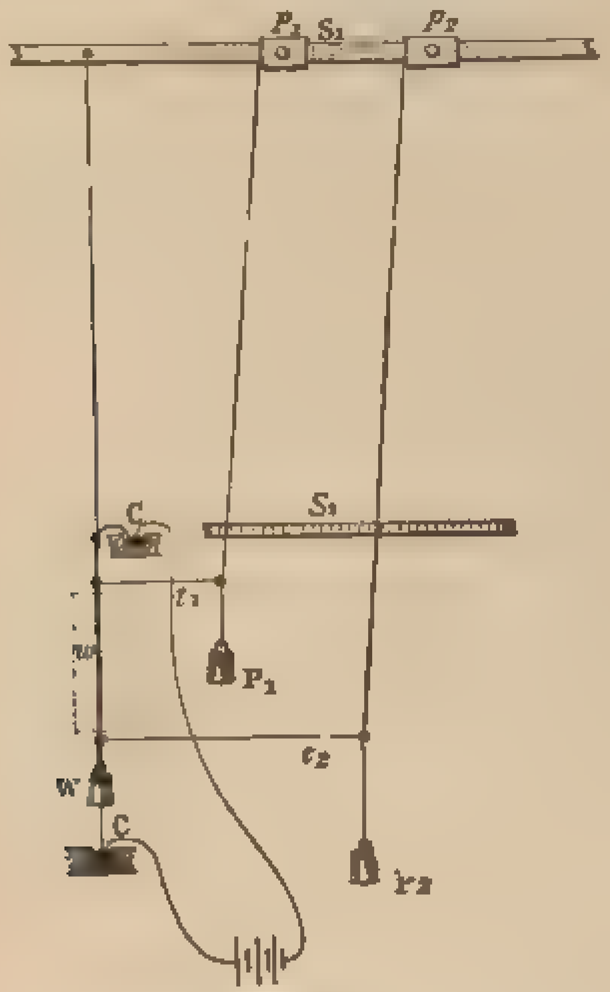


FIG. 151.

two sliding pieces which can be moved along a graduated cross-bar S_1 above, so placed that the ends are as nearly as possible in a plane parallel to the pole faces and passing through the middle of the space between them. The two pendulum threads and the wire, w, are thus nearly in one plane. One of these pendulums is made so long as to have its bob below the level of the

Pendulum Method of Measuring Small Forces.

lowest part of the pole faces, while the other has its bob a little below the level of the top of the pole faces, and the former is placed at the greater distance from the suspended wire. A thin thread attached to the upper end of the suspended wire is carried out horizontally and made fast at its other end to the suspension thread of the nearer pendulum.

A similar thread is attached at one end to a point of the wire near the bottom of the pole faces, and carried out similarly and made fast at the other end to a point nearly on the same level in the suspension thread of the further pendulum. The upper and lower ends of the wire, w, are placed, as shown, in mercury cups, to which are also connected the electrodes of a battery, by means of which a current can be sent through the wire w, and measured by means of a galvanometer in the circuit. A scale, S_2, is placed a little behind the plane of the threads, so that the position of a point in each, on the same level near their lower ends, can be easily read off.

When an experiment is made, the sliding pieces, p_1, p_2, are moved towards the left until the threads, t_1, t_2, are quite slack, and the positions of each thread on the upper and lower scales are read off and noted. The position of the wire, w, when t_1, t_2 are quite slack is also marked at the upper and lower ends of the pole faces or elsewhere. A current is then sent through the wire, w, in such a direction that the electromagnetic force acting on it moves it towards the left. The sliding pieces, p_1, p_2, are then moved towards the right so as to cause the pendulums to pull the wire by means of the threads t_1, t_2, back again to its initial position. When the upper and lower ends have come back to their former positions, the electromagnetic force on the wire is balanced by the pulls exerted by the pendulums. The positions of the pendulum threads are again read off on the upper and lower scales and noted, with the strength of the current flowing in w. From these results we can easily calculate the average intensity of the field at the place occupied by the wire, w. For let W be the mass of each of the pendulum bobs in grammes, d the distance through which the top of the pendulum thread has been carried by p_1 to the right of the point of the thread opposite to the lower scale S_2, d_2 the corresponding distance for the other pendulum, l the vertical distance between the levels of the tops of the pendulum threads and the lower scale measured in the same units as d_1, d_2, L the length of the opposed pole faces, and γ the strength of the current in C.G.S. units (one-tenth the number of amperes). The downward force in dynes on each of the masses is Wg, where g is the acceleration in centimetres per second per second ($=981\cdot4$ in latitude of

Calculation of Electromagnetic Force.

Glasgow) produced by gravity in a falling body at the place of experiment. The total pull to the right exerted by the threads on the wire is therefore $WG(d_1 + d_2)/l$, and this is equal to the pull towards the left on the wire produced by the electromagnetic action. If γ be the average intensity of the field along the wire in C.G.S. units, we have for this pull in dynes $I l g$. Hence we get the equation

$$IL\gamma = Wg\frac{d_1 + d_2}{l},$$

and therefore,

$$I = \frac{Wg}{L\gamma} \cdot \frac{d_1 + d_2}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Results of Actual Experiment.

In an experiment made on September 16, 1882, with a similar arrangement, W was 100 grammes, l 100 centimetres, C 188 in C.G.S. units of current, L 30 centimetres, and $d_1 + d_2$ 25·84 centimetres. Hence,

$$I = \frac{100 \times 981 \cdot 4}{30 \times 188} \cdot \frac{25 \cdot 84}{100} = 4496.$$

The wire, w, should not be so flexible as to bend perceptibly under the influence of the forces to which it is subjected, so that the value of I found may be nearly enough the average value of the intensity along a straight line in the space between the pole faces.

Method by Measurement of Couple on Circuit in Field.

In cases in which, as in many dynamo machines, the opposite pole faces of the electromagnets are at a considerable distance apart, with or without pieces of soft iron in the intermediate space, it is practically useful to find simultaneously the magnetic field intensity along two lines in the same plane, one in the vicinity of each pole face. This may be done by so placing the electromagnets that the two lines along which the field is measured are in a horizontal plane, and using, instead of the single wire carrying the current, a rectangle of copper wire or strip, of which the opposite sides are in these lines, supported on knife-edges in the bisecting line parallel to the pole face so that it can turn round that line as axis. The frame should be weighted symmetrically on the two sides of the line of knife-edges, so that it rests with just enough of stability in the horizontal position. The ends of the wire or strip forming the rectangle are brought out one above the other at one of the knife-edges with a piece of insulating material between them, and bent over so that the end of each dips into a mercury-cup

in line with the knife-edges. The electrodes of a battery are connected to the mercury cups and a measured current is sent round the rectangle. Since the poles have opposite magnetisms, the electromagnetic action causes one side of the rectangle to move upwards, the other side to move downwards, and thus turns the rectangle round the knife-edges.

Calculation of Electromagnetic Couple.

The moment of the electromagnetic forces is balanced by the action of weights, which may be riders of known weight made of wire, placed on the sides of the rectangle, which is thus brought back to its initial position. If we call I the average intensity of the fields along the two sides of the rectangle in the equilibrium position, and C the current strength, both as before measured in C.G.S. units, L the length of each side, and d the distance between them in centimetres, the moment of the electromagnetic forces round the knife-edges is $ICLd$. The opposite moment resisting the motion is, if only one weight of W grammes at a distance of d' cms. from the line of knife-edges is used, Wgd'. Hence, equating these moments, we get

$$I=\frac{Wg}{CL}\frac{d'}{d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

from which I can be calculated. If more than one weight, W, is used, each must be multiplied by its distance from the line of knife-edges, and the sum of the products multiplied by g for the equilibrating moment.

In some cases it may be convenient to use more than one turn of wire in the rectangle. If there be n turns, each of length L, nL is to be used instead of L in the formula above.

Circuit Suspended by Bifilar with Upper Ends Movable.

An obvious modification of this arrangement, which may be useful in some cases, is a rectangle suspended in a vertical plane, and kept in equilibrium in the proper position when no current is flowing through it, by means of a bifilar suspension, or a single thread or thin wire under torsion. When a current is sent through the frame, it is deflected round a vertical axis by the electromagnetic action, and is brought back to the initial position of equilibrium by means of two pendulums, the points of suspension of which are on sliding pieces which can be moved along horizontal parallel bars fixed above at right angle to the plane of the rectangle when in the equilibrium position, and in the same vertical planes as its sides. Each pendulum cord has attached to it a thread which pulls horizontally at the middle of one side of the rectangle. When the rectangle is deflected, the sliding pieces are moved in opposite directions, so that, in consequence of the opposite inclinations of the pendu-

lums to the vertical, forces restoring equilibrium are applied to the rectangle. As before, we have for the electromagnetic couple $I\gamma Ld$. Supposing the two points of suspension of the pendulums to be on one level, and the points of attachment of the pulling threads to the pendulum cords to be on a level lower by a distance of l cms., the distances of the verticals through the points of suspension from the corresponding verticals through the attachments of the threads to the pendulum cords to be d_1, d_2 cms. for the respective pendulums, and W grammes the mass of each bob, we have, for the moment of the equilibrating forces, the value $Wgd(d_1 + d_2)/l$.

Hence, equating moments, we get

$$I = \frac{Wg}{\gamma L} \cdot \frac{d_1 + d_2}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If IL is the same for both sides of the rectangle, d_1 and d_2 will be equal; but in general there will be a small difference between the two values.

Measurement of Electromagnetic Couple on Coil by Pendulum.

In some important practical cases the pole faces are of small area and are at only a small distance apart. If there is room, a small rectangular coil, similar to that of a siphon recorder (see Fig 152), but of comparatively few turns of wire, and without an iron core, may be hung, as described above, between the poles, with its plane parallel to the lines of force, by a bifilar or a torsion thread or wire, and a measured current sent through it. A rigid projecting arm fixed to the coil at the middle of its upper end and at right angles to the plane of the coil, has resting against it the suspension thread of a pendulum, attached at its upper end to a sliding piece movable along a horizontal bar carrying a millimetre scale, above and at right angles to the projecting arm; and by this means the coil is brought back to the initial position. When no current is flowing through the coil, the thread is allowed to hang vertically just touching the bar and the reading on the scale above noted. Let the difference between this reading and that obtained when the pendulum is deflected be d, and let l be the vertical height of the point of suspension above the projecting arm. The horizontal force exerted by the pendulum is $Wg\,d/l$, and the moment of this round the vertical axis about which the coil turns $Wgr.d/l$, where r is the distance of the pendulum thread from the central plane of the coil. If n be the number of turns in the coil, b cms. its mean breadth, and L cms. the mean length of each side, the

moment of the electromagnetic forces is $nbILy$. We have, therefore,

$$I = \frac{Wgrd}{nLybl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This method has frequently been used for the determination of the magnetic field intensity of the magnets of siphon recorders. The coil hanging in its place was used as the measuring coil, and when no current was flowing through it, was kept hanging vertically in stable equilibrium with its plane parallel to the lines of force by the bifilar threads attached beneath it. These threads were kept taut and bearing against the bridge B by the weights W, resting on a plane slightly inclined to the vertical. A current from one or two cells was then sent through the coil, and the difference of potential between the terminals of the coil measured by means of a potential galvanometer. The thread of the pendulum was made to pull against the projecting aluminium arm to which the siphon is attached as shown in the figure, so as to bring the coil back to the initial position. The value of d was then read off, and that of C deduced from the known resistance of the coil and the result of the measurement with the galvanometer, and being substituted with the values of the other quantities W, n, b, &c. in (4), gave the value of I.

FIG. 152.

Method by Damping of Coil Oscillating in Field. The field intensities of siphon recorders have sometimes been determined by the following method, which is interesting theoretically.

Advantage is taken of the signal-coil, which consists of a rectangular coil a little more than 5 cms. long and 2 cms. broad, made of thin wire and supported by a silk thread above, so as to hang in a vertical plane round

a rectangular core of iron, which nearly fills, but nowhere touches, the coil. To the lower end of the coil two silk threads are attached, as shown in Fig. 152, and are stretched against a bridge B by two weights resting on the inclined plane W. This bifilar arrangement gives a directive force, tending to bring the plane of the coil into parallelism with that of the bifilar threads; so that when the coil is disturbed from that position, which is one of stable equilibrium, and then left to itself, it will, if the circuit be not closed, vibrate about the position of equilibrium with a determinate period of oscillation, with slowly diminishing range until at last it comes to rest. But if the circuit be closed through a high resistance, the coil will come more rapidly to rest: and if we gradually diminish this resistance, deflecting the coil through the same angle and noting its subsidence at each diminution, we shall find it come more and more quickly to rest, until a resistance is obtained with which in circuit it just returns to the position of equilibrium without passing that position. When this resistance has been determined, the strength of the field can be calculated.

Determination of Critical Resistance of Coil for Oscillation.

Let θ be the deflection of the coil from the position of equilibrium at time t, and T its period of oscillation when the circuit is not closed. We have then, neglecting the resistance of the air and other disturbances, for the equation of motion,

$$\frac{d^2\theta}{dt^2}+\frac{4\pi^2}{T^2}\theta=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Theory of Method

Let now the circuit of the coil be closed; a retarding force due partly to air-resistance, but in the main to the current induced in the wire, and, if the effect of self-induction be neglected, proportional to the angular velocity, will act on the coil; and the equation of motion for this case will be of the form

$$\frac{d^2\theta}{dt^2}+k\frac{d\theta}{dt}+\frac{4\pi^2}{T^2}\theta=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For let I be the mean intensity of the magnetic field over the space occupied by the coil at time t, L the inductance of the circuit for that position of the coil, R the total resistance in the circuit, μ the moment of inertia of the coil round a vertical axis passing through its centre, l the effective length of wire in the coil (that is, the length of wire in its two vertical sides), and b the mean half-breadth of the coil. If we call N the number of

lines of force which pass through the coil at time t, and γ the strength of the induced current in the coil at that instant, we have plainly

$$N = bIl \sin\theta - L\gamma.$$

The rate at which N increases per unit of time is therefore

$$\frac{dN}{dt} = bIl \cos\theta \frac{d\theta}{dt} + bl \sin\theta \frac{dI}{dt} - \frac{d}{dt}(L\gamma);$$

and if θ be small, and I be therefore supposed the same for every position of the coil, we have approximately

$$\frac{dN}{dt} = bIl\frac{d\theta}{dt} - \frac{d}{dt}(L\gamma).$$

But dN/dt is the electromotive force due to the inductive action; hence the current γ is by Ohm's law given by the equation

$$\gamma = \frac{bIl}{R}\frac{d\theta}{dt} - \frac{1}{R}\frac{d}{dt}(L\gamma).$$

It was assumed that the second term of this expression for γ would prove negligible in comparison with the first, and this assumption was so far justified by the results of the experiments, which agreed fairly well with results obtained, for other instruments of the same pattern, by a modification of the second method described below.

The couple due to the action of the field on the current is $bIl\gamma$: and therefore, on the supposition of negligible self-induction, the retardation of the angular velocity of the coil at time t is

$$k\frac{d\theta}{dt} = \frac{b^2I^2l^2}{\mu R}\frac{d\theta}{dt}.$$

Hence (6) becomes

$$\frac{d^2\theta}{dt^2} + \frac{b^2I^2l^2}{\mu R}\frac{d\theta}{dt} + \frac{4\pi^2}{T^2}\theta = 0. \quad . \quad . \quad . \quad . \quad (7)$$

The motion represented by this differential equation will be oscillatory or non oscillatory, according as the roots of the auxiliary quadratic are imaginary or real—that is, according as $4\pi/T >$ or $< b^2 I^2 l^2/\mu R$. Hence, if R be the critical resistance at which the motion just ceases to be oscillatory, we have

$$I^2 = \frac{4\pi R\mu}{Tb^2l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

When l and b are expressed in centimetres, μ in grammes and centimetres, T in seconds, and R in cms. per second, I is given by this equation in absolute C.G.S. units of magnetic field intensity.

Mode of Experimenting.

The method of experimenting consisted in first finding the value of T, the free period of vibration of the coil with its circuit uncompleted, then finding the resistance which, being placed in circuit with the coil, just brought the needle to rest without oscillation. This resistance was conveniently obtained by means of a resistance-box included in the circuit, and therefore added no self-inductance to that in the coil. An aluminium arm attached to the coil, and carrying the siphon, served as an index to render the motions of the coil visible. The resistance R was first made much too great, so as to give a slow subsidence, then gradually diminished until the value which just prevented oscillation was reached; and it was found that this value could be determined easily within 50 ohms, and with great care, to 20 ohms. As the experiments on the recorders had to be made somewhat hurriedly, and disturbances generally were neglected, and, further, as μ was taken as equal to Wb^2, where W is the mass of the coil, the results could not be taken as giving more than a rough approximation to I: but those for two instruments are given below in illustration of the method. For both instruments the values of W, L, and b were the same, and were respectively taken as 3·343 grammes, 3338 cms., and ·95 cm. Each coil had a mean vertical length of 5 3 cm., a mean breadth of 1 9 cm., and contained 45 72 metres of fine wire arranged in 290 turns, and had a resistance of about 500 ohms.

Results of Actual Experiment.

	T.	R	I.
(1)	·465 sec.	3330×10^9 cms. per sec.	5150 C.G.S.
(2)	·500 ,,	3530×10^9 ,, ,,	5120 ,,

This method is obviously applicable in any case in which a coil can be suspended by a torsion wire, or bifilar, or other arrangement so as to have a measurable free period of vibration.*

* The method just described gives (theoretically) a means of determining the ohm. For suppose the coil hung in a sufficiently intense and uniform field, the intensity of which has been measured by another method, and the decrement of the oscillatory motion produced by the induction observed. Then the resistance could be calculated.

Method by Induced Currents.

The following method, which has been frequently used in the Physical Laboratory of the University of Glasgow, is very convenient and useful in many cases. It consists in exploring the magnetic field by means of the induced current in a wire moved quickly across the lines of force over a definite area in the field. The wire is in circuit with a reflecting "ballistic" galvanometer —that is, a galvanometer the system of needles of which has so great a moment of inertia that the whole induced current due to the motion of the wire has passed through the coil before the needle has been sensibly deflected. The deflection thus obtained is noted, and compared with the deflection obtained when, with the same or a smaller resistance in circuit, a portion of the conductor is made to sweep across the lines of force over a definite area of a uniform field of known intensity, such as that of the earth or its horizontal or vertical component.

In performing the experiments, it is necessary to take precautions to prevent any action except that between the definite area of the field selected and the wire cutting its lines of force. For this purpose the conducting-wire, which is covered with insulating material, is bent so as to form three sides of a rectangle, the middle one of which is of the length of the portion of field to be swept over. This side is placed along one side of the space over which it is about to be moved so that the connecting wires lie along the ends of the space; and the open rectangle is then moved in the direction of its two sides until the opposite side of the space is reached. The connecting wires thus do not cut the

lines of force, and the induced current is wholly due to the closed end of the rectangle.

Instead of a single wire cutting the lines of force, a coil of proper dimensions (for many purposes conveniently of rectangular shape), the mean area of which is exactly known, may be suspended in the field with its plane parallel to the lines of force, and turned quickly round through a measured angle of convenient amount not exceeding 90°; or it may be suspended with its plane at right angles to the lines of force and turned through an angle of 180°. If n be the number of turns, A their mean area, and I the mean intensity of the field over the area swept over in each case, then, in the first case, if θ be the angle turned through, the area swept over is $nA\sin\theta$ and the number of lines cut is $nIA\sin\theta$; in the second, the area is $2nA$, and the number of lines cut is $2nIA$. Suspended Coil.

In order that with the feeble intensity of the earth's field a sufficiently great deflection for comparison may be obtained, it is necessary that a relatively large area of the field should be swept over by the conductor. One convenient way is to mount on trunnions a coil of moderately fine wire of a considerable number of turns wound round a ring of large radius, like the coil of a standard tangent galvanometer, and arranged with stops so that it can be turned quickly round a horizontal axis through an exact half-turn, from a position in which its plane is exactly at right angles to the dip. This coil, if the ballistic galvanometer is sensitive enough, may always remain in the circuit. The change in the number of lines of force passing through the coil in the same direction relatively to the coil, produced by the half-turn, is plainly equal to twice as many times the area of the turn of mean area as there are turns in the coil (the effective area swept over) multiplied by the total intensity of the earth's magnetic force at the place of experiment. Or, and preferably when the horizontal component of the earth's magnetic force has been determined by experi- Earth Inductor.

ment, the coil may be placed in an east and west (magnetic) vertical plane, and turned through an exact half turn. The magnetic field intensity by which the effective area is to be multiplied is in this case the value of H.*

"Trapeze" Earth Inductor.

A sufficiently large area of the earth's field for comparison may, in some cases, be obtained very readily by carrying the wire along a rod of wood, say two or three metres long, and suspending this rod in a horizontal position by the continuations of the conductor at its ends from two fixed supports in a horizontal line at a distance apart equal to the length of the rod, and securing the remaining wires in circuit so that they may not cause disturbance by their accidental motion. The rod will thus be free to swing like a pendulum by the two suspending wires. The pendulum thus made is slowly deflected from the vertical until it rests against stops arranged to limit its motion. When the needle is at zero, the rod is quickly thrown to the other side against similar stops there, and caught. The straight conductor thus sweeps over an area of the vertical component of the earth's field equal to the product of the length of the rod into the horizontal distance between the two positions of the conductor at the extremities of its swing. The rod may be placed at any azimuth, as the suspending portions of the conductor in circuit, moving in vertical planes, can cut only the horizontal lines of force, and the induced currents thus produced have opposite directions and neutralize one another.

Theory of Method by Induced Currents.

The calculation of the results is very simple. By the theory of the ballistic galvanometer,† if q be the whole quantity of electricity which passes through the circuit, and if θ be the angle through which the needle has been deflected, or the "throw," we have, neglecting air resistance, &c.,

$$q = \frac{2}{G}\sqrt{\frac{\mu H}{m}} \sin \frac{\theta}{2}. \quad . \quad . \quad . \quad . \quad (9)$$

* The method of reducing results of observations to absolute measure by means of an earth inductor was used by Professor H. A. Rowland in his experiments on the magnetic permeability of iron, steel, and nickel.—*Phil. Mag.*, vol. 46, 1873.

† See p. 391 above.

where μ is the moment of inertia of the needle and attachments, m the magnetic moment of the needle, H the earth's horizontal magnetic force, and G the constant of the galvanometer. If θ be small, as it generally has been in these experiments, we have

$$q = \frac{1}{G}\sqrt{\frac{\mu H}{m}}\,\theta \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and the quantities of electricity produced by sweeping over two areas, A and A', are directly as the deflections.

Let A be the total area swept over in the field or portion of field the mean intensity I of which is being measured, A' and I' the same quantities for the known field, R, R' the respective total resistances in circuit, q, q' the quantities of electricity generated in the two cases, θ, θ' the corresponding deflections supposed both small; we have

$$q = \frac{AI}{R} = \frac{1}{G}\sqrt{\frac{\mu H}{m}}\,\theta,$$

$$q' = \frac{A'I'}{R'} = \frac{1}{G}\sqrt{\frac{\mu H}{m}}\,\theta'$$

and therefore

$$I = \frac{A'R\theta}{AR'\theta'}\,I' \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If convenient, θ and θ' may be taken as proportional to the number of divisions of the scale traversed by the spot of light in the two cases

The error caused by neglecting the effect of air

resistance, &c., in diminishing the deflection will be nearly eliminated if R and R' be chosen so that θ and θ' are nearly equal.

Solenoid Method of Reducing Ballistic Results to Absolute Measure.

The following method of reducing ballistic observations to absolute measure is very convenient when an earth inductor is not available. A short induction coil wound round the centre of an ordinary magnetizing helix, whose length is great compared with its diameter, is kept in circuit with the galvanometer. A measured current is sent through the wire of the helix, and when the needle is at rest the circuit of the helix is broken, and the galvanometer deflection read off. If N be the number of turns of wire per cm. on the helix, γ the current in electromagnetic C.G.S. units, the magnetic force within it is $4\pi N\gamma$ parallel to the axis; and if A' be the proper mean area of the cross-section of the helix, and n' the number of turns in the induction coil, the number of lines (unit tubes) of force passing out of the galvanometer circuit when the current is stopped is $4\pi Nn'A'\gamma$. [See (75) p. 284, above.] Hence R' denoting the total resistance in circuit, the total quantity q' of electricity generated is $4\pi Nn'A'\gamma/R'$ and instead of (11) we get

$$I = 4\pi Nn'\gamma \frac{A'R\theta}{AR'\theta'} \quad . \quad . \quad . \quad . \quad (12)$$

Section II

DETERMINATION OF MAGNETIC DISTRIBUTION, MAGNETIC INDUCTION, AND PERMEABILITY

The ballistic method of investigation was also used by Prof. H. A. Rowland* for the determination of that ideal surface distribution of magnetism on magnets which, as shown by Gauss, may be supposed to replace the actual distribution so far as the production of the external field is concerned. Magnetic Distribution Ballistic Method of Investigation.

That such a distribution is possible and determinate is obvious from the electric analogue. Consider a distribution of electric potential corresponding precisely to the given one of magnetic potential, and produced by a volume distribution of equal quantities of positive and negative electricity, corresponding in position to the magnet. This is obviously possible. Then suppose a thin conducting surface to be placed round the electric distribution, corresponding exactly to the surface of the magnet, and connected to the earth. The potential at external points is thereby reduced to zero. Thus the induced distribution produces a distribution of potential equal and opposite to that due to the electric system within the shell, and therefore if reversed would produce precisely the same distribution of potential as the latter does. The total amount of this induced distribution is equal and opposite to that in the internal system, and the distribution is, as we have seen in vol. I., uniquely determinate. Hence translating back to the magnetic case, it is clear that a distribution of magnetism over the surface of the magnet may be supposed to exist and produce the external field, and may be made the subject of experimental research. Ideal Surface Distribution on a Magnet

When found it expresses the mode in which the lines of induction enter or leave the surface of the magnet; but it is not to be taken as having a real existence.

* *Phil. Mag.* vol. i., 1875.

Rowland's Method of Experimenting.

Rowland used a thin ring of wire made just large enough to pass round the magnet experimented on, and placed in circuit with a ballistic galvanometer. It was then, while encircling the magnet and held with its plane at right angles to the axis of the magnet, slided quickly along the magnet through each of a succession of equal short distances, and the deflection of the needle noted for each motion. The deflections thus obtained gave for their magnets an approximate estimate of the density at different points along the magnet, of the ideal surface distribution, and the results were reduced to absolute measure by means of an earth inductor.

Determination of Pole-Strength of Magnet.

This method with any convenient method of reduction of results to absolute measure (for example, the helix arrangement just described) gives a very ready means of estimating with much exactness the total quantity of magnetism according to the ideal distribution in one pole or one end of a magnet, whether of bar, horse-shoe, or other shape. The ring, which for the present purpose may be larger, and thick enough to contain any convenient number of turns, is placed at the central or nearly neutral region of the magnet and then quickly pulled off and away from the magnet, and the galvanometer deflection, θ, noted. A measured current is then sent through the helix, and the deflection θ', produced by suddenly opening the circuit of the helix, also observed.

Theory of Method.

Let n be the number of turns in the ring of wire, ϕ the total quantity in C.G.S. units of magnetism in the portion of the magnet swept over, then the number of

lines of induction cut through by each turn of wire in the ring is $4\pi\phi$, and if R be the total resistance in circuit, the total quantity of electricity generated is $4\pi n\phi/R$. We have therefore

$$q = \frac{4\pi n\phi}{R} = \frac{HT}{\pi G}\sin\frac{\theta}{2}$$

and for the helix we get from the calculation above

$$q' = \frac{4\pi Nn'A'\gamma}{R'} = \frac{HT}{\pi G}\sin\frac{\theta'}{2}.$$

By division we find

$$\phi = NA'\gamma\frac{n'R\sin\dfrac{\theta}{2}}{nR'\sin\dfrac{\theta'}{2}} \quad . \quad . \quad . \quad . \quad (13)$$

and if the deflections are small angles

$$\phi = NA'\gamma\frac{n'R\theta}{nR'\theta'} \quad . \quad . \quad . \quad . \quad . \quad (13')$$

Calculation of Density of Ideal Distribution.

This equation is of course also applicable to the reduction to absolute measure of the results of determinations of magnetic distribution made by the ballistic method. The value of ϕ deduced for each deflection divided by the area of the correspondingly small portion of the magnet is approximately the surface density of the ideal distribution, the distribution on the end faces being of course included in the end deflections.

Determination of Total Magnetic Induction.

The ballistic method has been used by Rowland, Thomson, Hopkinson, Ewing, and others for the investigation of the magnetic properties of iron. We give here some examples of its use for the determination of total magnetic induction in a specimen of iron. Let for example the specimen tested be an iron rod or wire magnetized by a helix of wire in which a current flows. An induction coil of a suitable number of turns encircles the magnetizing coil midway between the ends, and is in circuit with a ballistic galvanometer. If the magnetizing current be altered there will in general be a change of magnetic induction. An induced current is thereby made to flow in the ballistic galvanometer circuit, the corresponding deflection is observed, and the change of induction calculated from its amount. Thus if θ be the angular deflection corrected if necessary for damping (see pp. 394, 486 above) and n be the number of turns in the induction coil, the total induction through each turn is given by the equation

$$\frac{n}{R}\, d\mathsf{B} = \frac{HT}{\pi G} \sin\frac{\theta}{2}$$

or

$$d\mathsf{B} = \frac{R}{n}\frac{HT}{\pi G} \sin\frac{\theta}{2} \quad . \quad . \quad . \quad . \quad (14)$$

The determination of H, T, and G may be avoided by the use of an earth-inductor or helix as already described: so that

$$q' = \frac{4\pi N n' A'\gamma}{R'} = \frac{HT}{\pi G} \sin\frac{\theta'}{2}$$

and

$$d\mathbf{B} = 4\pi N A'\gamma \frac{n'R \sin\frac{\theta}{2}}{nR' \sin\frac{\theta'}{2}} \quad . \quad . \quad . \quad (15)$$

Ballistic Method of Determining Magnetic Induction.

By beginning from zero current and increasing the current by small steps observing the increment $d\mathbf{B}$ of induction at each, the total induction at any stage can be obtained by addition of the previous increments,

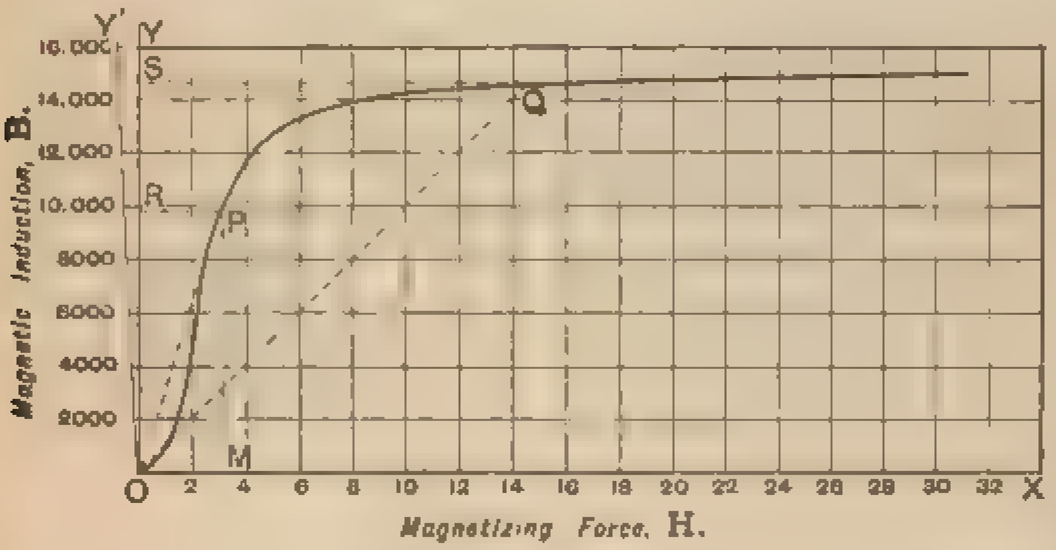


Fig. 153.

from zero. The results can be represented in a curve, with inductions as ordinates and strengths of magnetizing current as abscissæ. The abscissæ will thus (with a certain correction specified below) be proportional to the values of **H**, the intensity of the magnetizing field at any point, for the different strengths of current. Such a curve is shown in Fig. 153.

Arrangement of Apparatus for Ballistic Experiments. The general arrangement of apparatus for ballistic experiments is shown in Fig. 154, which is taken from a paper by Dr. J. Hopkinson on the Magnetization of Iron.* A magnetizing coil C, C, surrounds a rod of iron shown as B, B', in Fig. 156. In circuit with CC is a battery E the current from which can be varied by the liquid rheostat, F, and measured by the current-meter G. The magnetizing coil is in two halves and between

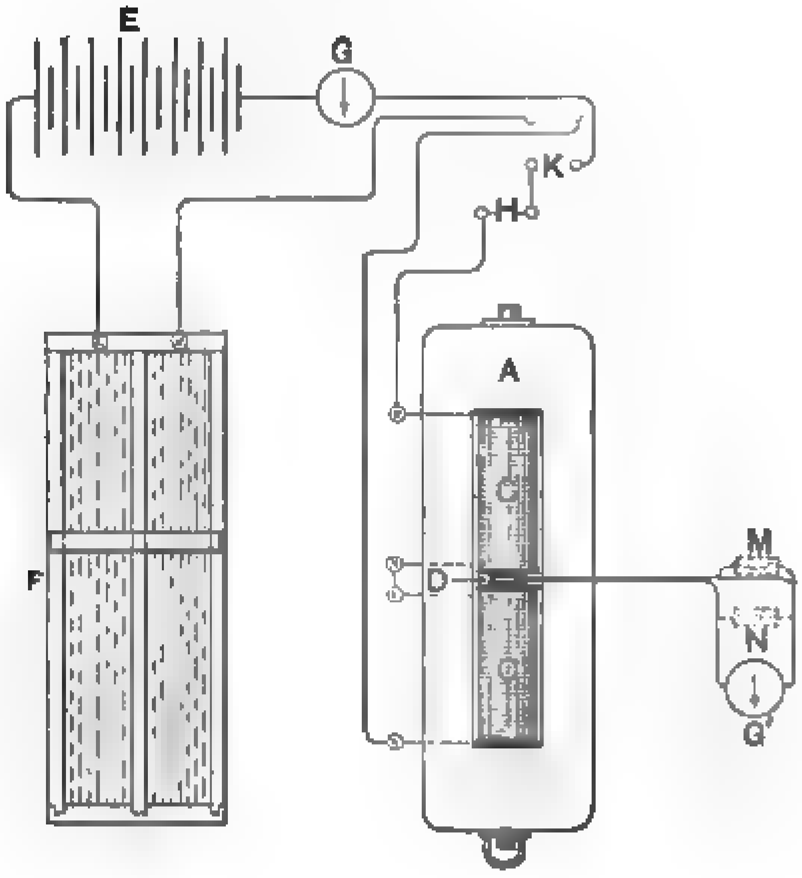


Fig. 154.

them surrounding the bar is a small induction coil D, which is in circuit with the ballistic galvanometer G', through a proper key. H is a simple make and break arrangement, K a current reverser. At M additional resistance can be placed in the ballistic circuit, and at N the galvanometer can be shunted.

* *Phil. Trans. R. S.*, Part II., 1885.

Hopkinson's Experiments.

By such an arrangement it is possible to put the magnetizing current through any series of changes, for example to increase it by aid of the rheostat from zero to any required positive value, gradually diminish it again to zero, then reverse the current, gradually increase its negative value and finally diminish it again to zero.

The arrangement here figured may of course be varied to suit any particular case. In Dr. Hopkinson's experiments the bar B, B', was in two halves, of which one B could be suddenly withdrawn by the handle shown in Fig. 155. When this was done the current in the magnetizing circuit was simultaneously broken and the coil D, which was attached to a spring, was pulled suddenly out of the field. Thus at any time a reading

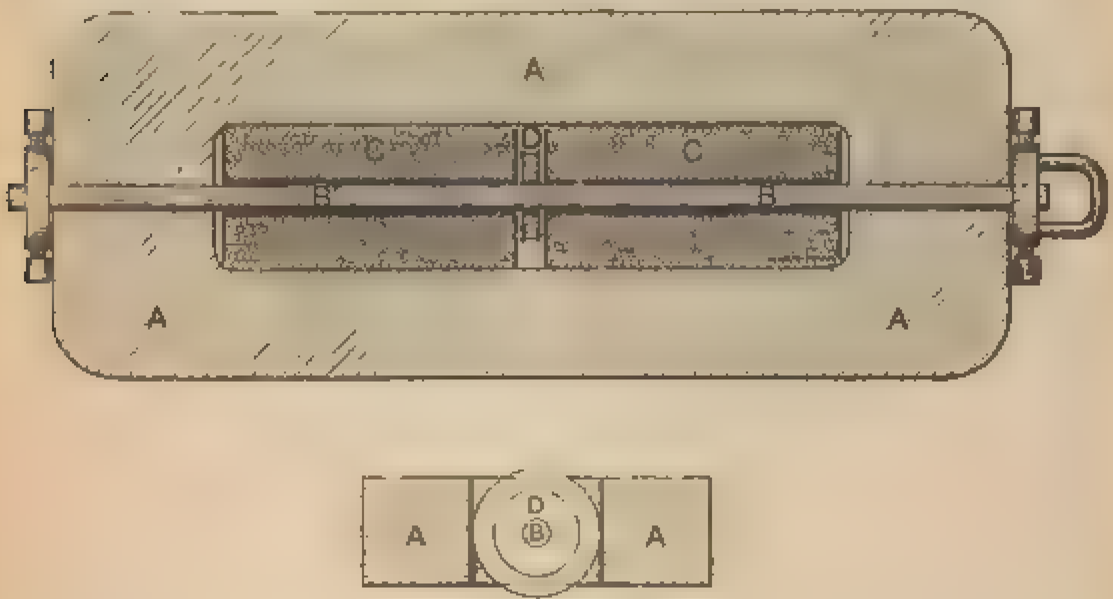


FIG. 155.

could be obtained for the whole induction in the coil; and the procedure adopted was to subject the bar to successive specified series of changes of magnetizing force, and test the final state of the bar in the manner just indicated. A correction was made for the excess of induction due to the fact that the area of the induction coil was greater than that of cross section of the bar. This correction was found by substituting a rod of copper and a rod of wood for the iron, and measuring the induction with a considerable magnetizing force applied.

The bar had its ends embedded in a mass of soft iron, A, A, the action of which will be discussed presently. Some of the

principal results of these experiments will be found below (p. 731).

Correction for Induction between Iron and Secondary Coil.

In general there is included in the induction through the induction coil a certain part over and above the induction in the iron, namely, the part of the induction which passes between the iron and the coil, or in the space occupied by the layers of wire. As a rule this will be slight if the iron nearly fills the magnetizing coil, and it might be avoided by winding the induction coil under the magnetizing coil, and close round the iron; but very frequently it is desirable to be able at any time to withdraw the iron from the magnetizing coil, or suddenly to slip the magnetizing coil completely off, and this renders the outside position of the induction coil in general more convenient. The correction amounts to $(A' - A)\mathbf{H}$, if A' be the mean area of cross-section of the magnetizing coil, A that of the iron, and $\mathbf{H}$ the field intensity produced within the coil by the current. Thus if B denote the integral induction $\left(\int \mathbf{B} dS\right)$ through each turn of the induction coil, the part of this which exists in the iron is $\mathsf{B} - (A' - A)\mathbf{H}$

Effects of Ends of Helix and of Bar

We have here supposed $\mathbf{H}$ to be constant, but this will not be practically the case unless the specimen be a very long straight bar magnetized by a long helix. In this case at points within the helix at a considerable distance from the ends, the value of the part of $\mathbf{H}$ due to the current may, as we have seen above (p. 261), be taken as constant and equal to $4\pi n\gamma$, where n is the number of turns of wire per unit of length on the helix. If necessary the ends of the helix can be taken into account as shown at p. 260. There remains however even when this part of $\mathbf{H}$ is constant from point to point an allowance to be made for the magnetic force produced by the magnetization of the iron itself. This amounts in a long bar uniformly magnetized simply to a correction for the ends, and is a demagnetizing force, or force opposed to the other part of $\mathbf{H}$. In short we have if $\mathbf{H}$ be the field intensity at any point

Demagnetizing Force in Prolate Ellipsoid

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$$

where $\mathbf{H}_1$ is due to the current and $\mathbf{H}_2$ to the magnetization of the iron. Also

$$\mathbf{H} = 4\pi n\gamma + \mathbf{H}'_1 + \mathbf{H}_2 \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where $\mathbf{H}'_1$ denotes the effect of the ends of the helix, and is very small in general.

To find a superior limit for $\mathbf{H}_2$ for a long bar we may consider the bar as a very prolate ellipsoid. Then by (100), p. 54, we

have for the field intensity within the bar due to its own magnetization supposed uniform and of intensity $\mathbf{I}$

$$\mathbf{H}_2 = -L\mathbf{I} \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

where L has the value given in (101), p. 54. If $(\mathbf{B} - \mathbf{H})/4\pi$ be put for $\mathbf{I}$,

$$\mathbf{H}_2 = -L\frac{\mathbf{B} - \mathbf{H}}{4\pi} \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

or if $\mathbf{B}$ be very great in comparison with $\mathbf{H}$ as is generally the case in soft iron

$$\mathbf{H}_2 = -\frac{L}{4\pi}\mathbf{B} \quad . \quad . \quad . \quad . \quad . \qquad (18')$$

Table for Calculation of Demagnetizing Force in Ellipsoid.

Values of L and $L/4\pi$ calculated from the expression (101) have been given by Prof. Ewing for different ratios of length of bar to diameter of cross-section, that is different values of $1/\sqrt{1-e^2}$, where e is the eccentricity of the ellipsoid. These are given in the following table, with an additional number for the ratio 1000.

Ratio of Length to Diameter of Cross-section.	L	$\frac{L}{4\pi}$
50	·01817	·001446
100	·00540	·000430
200	·00157	·000125
300	·00075	·000060
400	·00045	·000037
500	·00030	·000024
1000	·000089	·000007

Demagnetizing Force in Sphere or Disk.

It is instructive to compare these values of the coefficient L with the corresponding quantities for a uniformly magnetized sphere or a very oblate ellipsoid of revolution, a disk in fact. For the sphere the value (p. 55) is $\frac{4}{3}\pi$, for the disk 4π.

Ratio of Actual to Applied Magnetizing Force in Prolate Ellipsoid: The ratio of the magnetizing force $\mathbf{H}$ actually existing at any point where the magnetization is of intensity $\mathbf{I}$, to the magnetizing force $\mathbf{H}_1$ applied by the coil is

$$\frac{\mathbf{H}}{\mathbf{H}_1} = 1 - \frac{L\mathbf{I}}{\mathbf{H}_1} = 1 \quad . \quad . \qquad (19)$$

if κ be the magnetic susceptibility. Thus

$$\frac{\mathbf{H}}{\mathbf{H}_1} = \frac{1}{1 + L\kappa} \quad . \quad . \quad . \quad . \quad . \qquad (19')$$

Examples. If κ be, say, 200 and the length be 100 times the diameter, the value ·00540 of L derived from the above table gives

$$\frac{\mathbf{H}}{\mathbf{H}_1} = \frac{1}{2{\cdot}08}$$

or the demagnetizing force is about $\frac{1}{2}$ that applied by the coil.

For the ratio 500 and 1000 of length to diameter the values of $\mathbf{H}/\mathbf{H}_1$ are respectively 1·314, 1·0445. The demagnetizing force is in the former case equal to about $\frac{1}{3}$, and in the latter about ·045 of the magnetizing force actually operative.

Graphical Method of Correcting for Effect of Ends of Long Bar The values of the induction thus found for different field intensities applied by the current therefore correspond, to smaller magnetizing forces than those directly calculated from the current, in the ratio of $\mathbf{H}_1/(\mathbf{H}_1+\mathbf{H}_2)$. In the graphical representation of the results of experiments this can be corrected for very easily by drawing a line OY' (Fig. 157) inclined to the left of the axis of OY at an angle $YOY' = \tan^{-1}(L/4\pi)$ and measuring the values of $\mathbf{H}_1$ from this line, instead of form OY. For we have

$$\mathbf{H} = \mathbf{H}_1 - L\mathbf{I} = \mathbf{H} - \frac{L}{4\pi}\mathbf{B} \quad . \quad . \quad . \quad . \qquad (20)$$

nearly, that is any induction $\mathbf{B}$ corresponds to an operative magnetizing force less than $\mathbf{H}_1$ by the fraction $L/4\pi$ of $\mathbf{B}$. By the construction given therefore the points in the curve are laid down at once in their correct positions.

Endless Bar: Anchor Ring. The condition of endlessness may be attained by the use of an anchor ring of the material and wrapping it round uniformly with wire, but in this case the field, as shown at p. 279 above, is not the same at all distances from the axis. In fact if N be

the total numbers of turns in the magnetizing helix, γ the current in each, $\mathbf{H}$ the force at a point within the core at distance r from the centre, $2\pi r\mathbf{H} = 4\pi N\gamma$ that is

Field Within Anchor Ring.

$$\mathbf{H} = \frac{2N\gamma}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

or if n be the number of turns per unit length of the circular axis of radius r_a

$$\mathbf{H} = 4\pi n\gamma \frac{r_a}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and r_a may be made as nearly equal to r as we please by diminishing the dimensions of cross-section relatively to r_a.

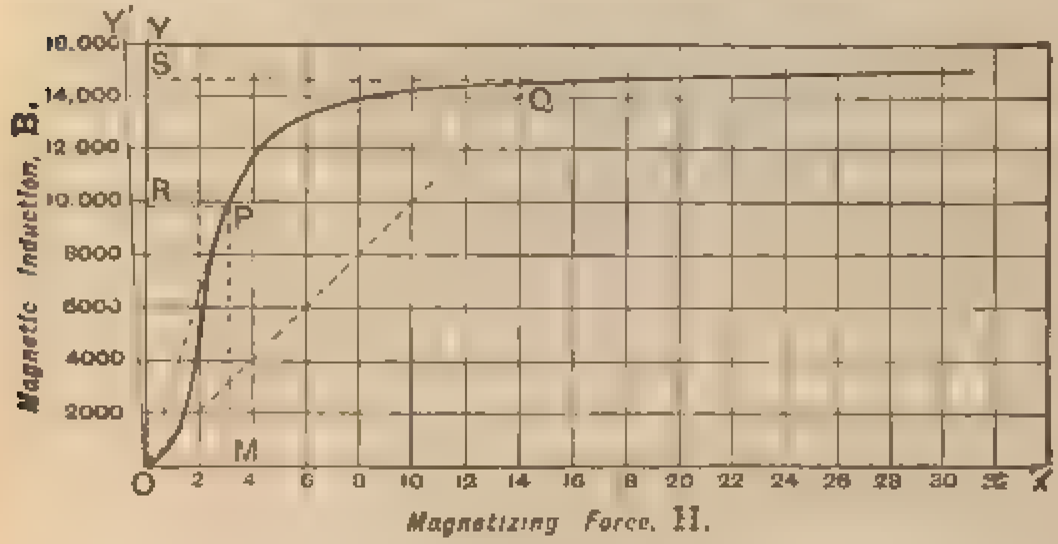


FIG. 156.

Experiments on anchor rings have been made by Rowland *, Bosanquet †, Ewing ‡ and others. Of course in such experiments the ballistic method is the only one that can be followed.

Bar with Soft Iron Yoke.

Practically, endlessness may be attained by placing the bar to be experimented on in a space cut in a block of as permeable iron as possible, so that the ends of the bar fit deeply and closely

* *Phil. Mag.* Aug. 1873, and Nov. 1874.
† *Ibid.*, Feb. and May 1885.
‡ *Phil. Trans. R. S.*, 1885, or *Magnetic Induction in Iron and other Metals*, p. 70.

into sockets in the end faces of the space, and the magnetic circuit is completed by the mass of iron so that no lines of induction escape into the medium outside the iron. This method is due to Dr. J. Hopkinson, who has employed it in ballistic experiments on the magnetic properties of iron rods.

Application of Principle of Magnetic Circuit.

How far endlessness can be attained in this way can be estimated very conveniently by means of the idea of the magnetic circuit referred to at p. 281 above. It has been shown that the integral induction across a surface cutting all the tubes of induction is

$$\mathbf{B} = \frac{4\pi N\gamma}{U\int\frac{ds}{\mu dS}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where dS is the area of a tube of induction at any point where the permeability is μ, ds an element of distance along the tube N the number of turns of wire carrying current γ, and traversed by each line, and U the ratio of $\mathbf{B}ds$ to $\mathbf{B}$. The numerator on the right of (23) is what Bosanquet* has called the magnetomotive force of the magnetic circuit, the denominator the magnetic resistance (or *reluctance*, as proposed by Heaviside).† In the specimen rod, if its cross-section is S, and length l we have from (23)

$$\mathbf{B} = \frac{4\pi N\gamma}{\frac{l}{\mu S} + U\int\frac{ds}{\mu' dS}}$$

where the integral in the denominator refers only to the portion of the circuit formed by the yoke; the first term is obviously the value of the integral for the specimen. If we consider the iron yoke as throughout of uniform cross-section S', length l' and uniform permeability μ' we get

Magnetic Circuit of Bar and Yoke.

$$\mathbf{B} = \frac{4\pi N\gamma}{\frac{l}{\mu S} + \frac{l'}{\mu' S'}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

* *Phil. Mag.* March, 1883

† The term *ampere-turns* is now very generally employed in practical work to designate $10N\gamma$, that is the product of the number of turns and the current in amperes. The magnetomotive force is thus $\cdot 4\pi$ times the ampere turns given by the magnetizing coil.

Thus

$$\mathbf{H} = \frac{\mathbf{B}}{\mu S} = \frac{4\pi N\gamma}{l + l'\frac{\mu S}{\mu' S'}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

or **H** in the iron is less than $4\pi N\gamma/l$, which would be the value if the rod were infinitely long, by the amount $4\pi N\gamma l'\mu S/\mu' S'$. Thus if μ' is great in comparison with μ, the correction can be made very small.

There is a further correction for the passage of the lines of induction to the iron yoke. For one thing the effective size of the iron at the junction is not equal to the cross-section of the iron yoke, though this may be taken account of by an addition to l'. If the bar were welded to the yoke, or formed one piece with it, the proper correction would be that applicable to a conducting wire joining two large masses of metal. The joint however between the specimen and the yoke has quite a perceptible effect and is in fact equivalent to a narrow air-gap.

Air-gap in a Magnetic Circuit.

The effect of an air-gap in a magnetic circuit may be studied by calculating it for the case of a ring split by a narrow gap. There is continuity of the induction in the iron and in the air on the two sides of the surface of separation, and if the gap be narrow very few even of the lines near the edges will spread out laterally before again entering the iron. We may take therefore the total cross-section of the induction tubes in the gap as equal to that of the iron ring. Thus taking the total induction and putting x for the width of the gap we get

$$\mathsf{B} = \frac{4\pi N\gamma}{\frac{1}{S}\left(\frac{l}{\mu} + x\right)}$$

and

$$\mathbf{B} = \frac{4\pi N\gamma}{\frac{l}{\mu} + x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

if the induction **B** be taken as having the same value at every point.

If x were zero we should have

$$\mathbf{B} = \frac{4\pi N\gamma}{\frac{l}{\mu}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Iron Equivalent of Air-gap.

The width x of the air-gap is thus equivalent to a length μx of iron, and therefore cutting out a slice of thickness x from an iron circuit is equivalent to increasing the length of the iron circuit by an amount $(\mu - 1)x$.

If l/μ may be taken as small in comparison with x, as in the case of some forms of electromagnet, notably certain "large surface" horse-shoes made by Dr. Joule, (27) becomes

$$\mathbf{B} = \frac{4\pi N\gamma}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27')$$

Maximum Field producible between Close Pole Faces by Given Current.

This by the continuity of induction is the field intensity between the poles, and gives a convenient rule for calculating the field intensity between the poles of a horse-shoe electromagnet with close pole faces of considerable surface, and a short iron circuit. It shows that even if the iron were of infinite permeability the field intensity due to a current γ could not exceed $4\pi N\gamma/x$. The iron should in such a case as this be considerably below saturation, otherwise the permeability would be low, and the resistance of the iron part of the circuit sensible.

Du Bois (*Phil. Mag.* Nov. 1890) has compared the effect of an air-gap with that of the ends of an ellipsoid. The force operative in the iron is nearly uniform, and has the value

Effect of Air-gap Compared with that of Ends of Ellipsoid.

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} = \frac{4\pi N\gamma}{l + \mu x}.$$

But the force applied by the current is

$$\mathbf{H}_1 = \frac{4\pi N\gamma}{l}$$

and therefore

$$\mathbf{H} = \mathbf{H}_1 - 4\pi N\gamma \frac{\mu x}{l^2 + \mu l x} = \mathbf{H}_1 - \mathbf{H}_1 \frac{\mu x}{l + \mu x}.$$

But $\mu = 1 + 4\pi\kappa$, if κ be the susceptibility. Hence

$$\mathbf{H} = \mathbf{H}_1 \left\{ 1 - \frac{(1 + 4\pi\kappa)x}{l + (1 + 4\pi\kappa)x} \right\} = \mathbf{H}_1 \left(1 - \frac{4\pi\kappa x}{l + 4\pi\kappa x} \right)$$

approximately. But this gives

$$\mathbf{H} = \mathbf{H}_1 - \frac{4\pi x}{l} \kappa \mathbf{H} = \mathbf{H}_1 - \frac{4\pi x}{l} \mathbf{I} \quad . \quad . \quad . \quad (28)$$

Comparing this with the case of an ellipsoid discussed above we see that $4\pi x/l$ takes the place of L, and that therefore the ring with air-gap is equivalent to an ellipsoid with this value of L.

Effect of Air-gap in Diminishing Residual Magnetism.

It is clear that when an air-gap is made in a ring the residual magnetism must be much less than when the ring is whole. For if $\mathbf{I}_r$ be the intensity of the residual magnetism at any instant, a demagnetizing force is operative of amount $4\pi x \mathbf{I}_r/l$.

Effect of Joint on Magnetic Resistance of Bar.

The effect of a joint on lengthening the magnetic circuit is easily estimated. The induction produced by a succession of different values of $\mathbf{H}$ is first observed so that a curve showing the results can be drawn, then a joint is made, and the forces again applied and the induction observed. Then for the uncut bar

$$\frac{\mathbf{B}}{\mu} = \frac{4\pi N\gamma}{l}$$

when the bar is cut we have

$$\frac{\mathbf{B}'}{\mu'} = \frac{4\pi N\gamma}{l + \mu' x}$$

Width of Air-gap equivalent to Joint.

where x is the width of the equivalent air-gap. Thus

$$\frac{\mathbf{B}'}{\mathbf{B}} = \frac{\mu'}{\mu}\,\frac{l}{l + \mu' x}$$

or

$$x = \frac{l}{\mathbf{B}'}\left(\frac{\mathbf{B}}{\mu} - \frac{\mathbf{B}'}{\mu'}\right) \quad . \quad . \quad . \quad (29)$$

Thus x is expressed in terms of the inductions produced by the same magnetizing force $\mathbf{H}$, applied by the current. This formula is well adapted for finding x from tables of results.

An equivalent expression in terms of the forces applied by the coil to produce the same induction, when the bar is cut and uncut, is more convenient when the value of x is to be obtained graphically. When the bar is uncut

$$\frac{\mathbf{B}}{\mu} = \frac{4\pi N\gamma}{l} = \mathbf{H}_1.$$

After the bar is cut

$$\frac{\mathbf{B}}{\mu} = \frac{4\pi N\gamma'}{l + \mu x} = \mathbf{H}_1'\frac{l}{l + \mu x}.$$

Hence

$$x = \frac{\mathbf{H}_1' - \mathbf{H}_1}{\mu \mathbf{H}_1} l = \frac{\mathbf{H}_1' - \mathbf{H}_1}{\mathbf{B}} l \quad . \quad . \quad . \quad . \quad (30)$$

Thus the value of x is to be found by laying down for the two cases curves with inductions as ordinates and the values of the magnetizing forces applied by the coil as abscissæ, and measuring the difference between the abscissæ for which the

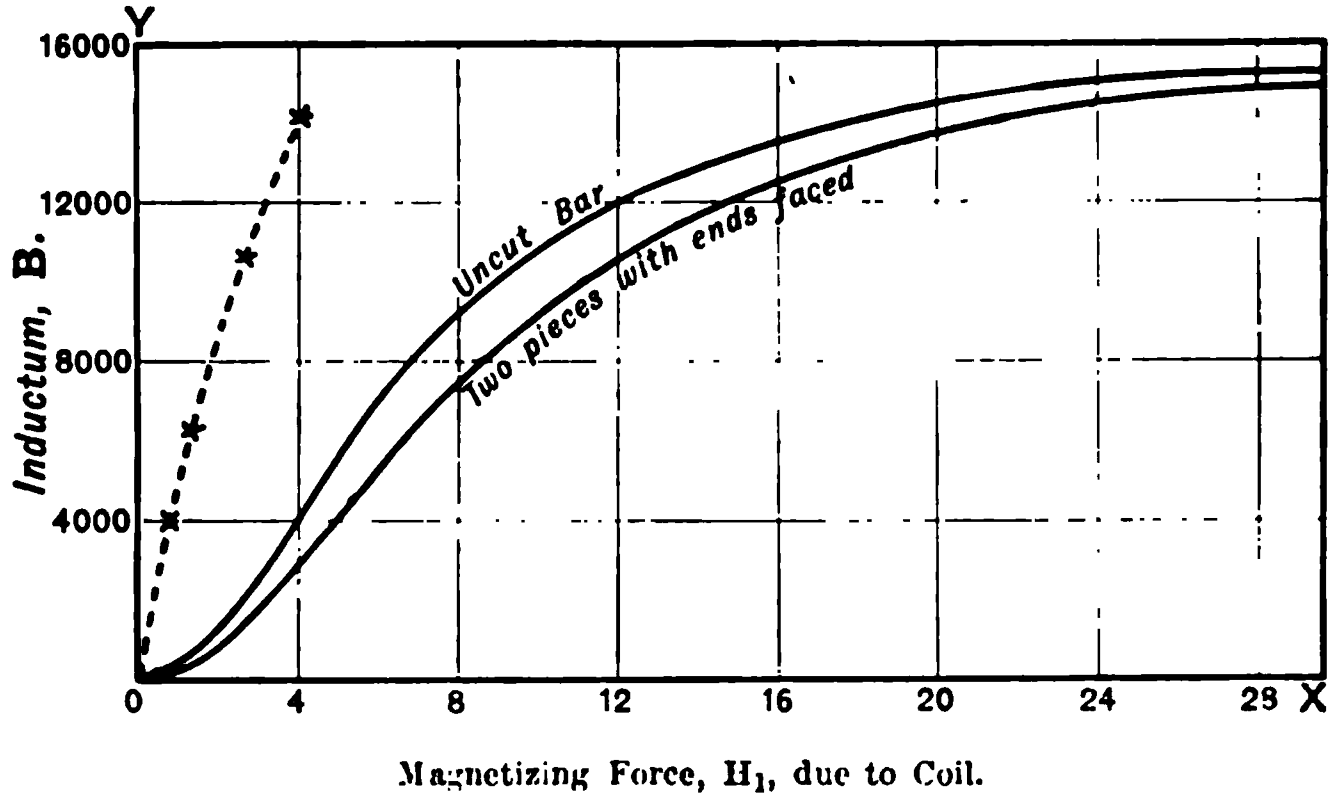


Fig. 157.

inductions are the same. This gives $(\mathbf{H}_1' - \mathbf{H}_1)/\mathbf{B}$, and so x can be calculated at once.

Observed Effect of Joint on Iron.

The following are results obtained by Ewing for an iron bar tested first solid, then after having been cut into parts the adjacent ends of which are worked into true planes and placed in contact. Fig. 157 shows the induction curves in the two cases, and by the dotted curve in the left values of $\mathbf{H}_1' - \mathbf{H}$ for different values of $\mathbf{B}$. The values of x are given in the following table :—

Induction B	Width of equivalent air gap, x, in cms.
4000	·0026
6000	·0030
8000	·0031
10000	·0031
12000	·0035
14000	·0037

Pressing Faces of Joints together Reduces Resistance.

It will be noticed that the dotted curve is somewhat convex towards the axis OY. In another of Ewing's experiments, however, this curve was slightly concave to OY, so that it seems likely that the width of the gap is in general nearly independent of the value of the induction.

It was found in these experiments, that the application of a

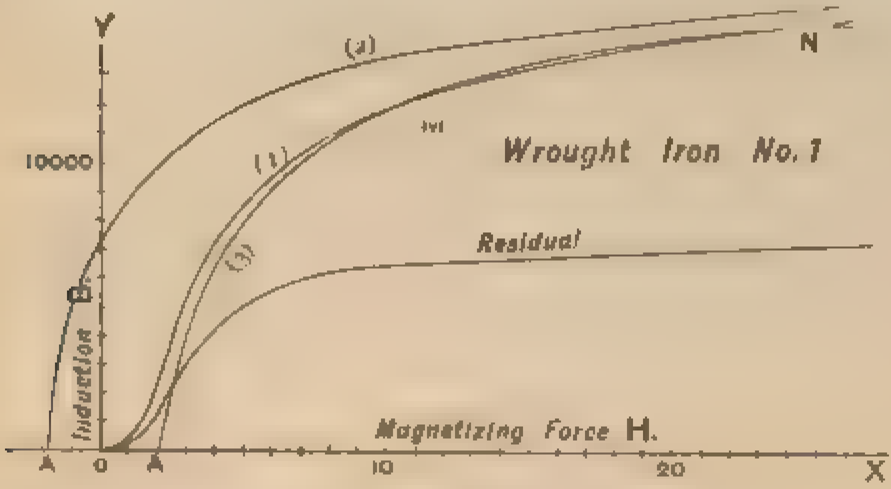


FIG. 158.

force of 226 kilogrammes weight per sq. cm., pressing the end faces of the bar at the joint together, even annulled its effect completely for a magnetizing force of 5 C.G.S., but not so perfectly when the magnetizing forces were higher.

Hopkinson's Experiments on Wrought Iron

Fig. 158 shows some of the results obtained by Dr. Hopkinson for wrought iron. Only a part of the curves, which extended to magnetizing forces of over 200 units, is

given ; but for the higher forces the curves only became more nearly parallel to the axis of abscissæ. The curve marked (1) shows the induction increasing from zero with **H** ; that marked (2) is the descending curve obtained with diminishing values of the magnetizing force. The diagram shows part of the results for a cycle of changes of magnetizing forces similar to that described at p. 212 above. The curves below the axis OX are not given, as they consist very nearly of a repetition of the curves in the diagram, obtained by turning the figure round the axis OY and then round OX. The magnetizing force was gradually increased from zero (the iron being initially without magnetization) to a high positive value, then diminished through zero to a considerable negative value, and finally increased by gradual steps to as large a positive value as before. The curve marked (3) is that given after the second passage of the induction through zero, that is from a negative to a positive value. Curves (1) and (3) do not approach one another quite closely for the highest magnetizing forces ; but (3) crosses (1) twice at M and N, which show that the negative magnetization which the bar had received rendered it more difficult to magnetize positively by small or high forces, but distinctly more easily magnetized by forces of intermediate amount. This has frequently been noticed both with iron and steel. Curves (2) and (3) with the continuations below OX and to the left of OY however form a nearly closed loop, the area of which is roughly $4 \times OA \times$ maximum induction. Areas $\div 4\pi$ of such cycles for different samples of iron and steel are given in the following table. They represent, as we shall see presently, the energy dissipated in heat in unit volume of the iron during the cycle of magnetization.

Cycle of Magnetization.

Areas of Cycles for Different Materials.

Description of Specimen.	$\dfrac{\text{Area from curve}}{4\pi}$	$\dfrac{4 \times \text{OA} \times \text{max. induction}}{4\pi}$
Wrought Iron (annealed)	17247	13356
Grey Cast Iron	15139	13037
Whitworth Mild Steel—		
(annealed) .	45903	40120
(oil-hardened)	61898	65786
(annealed) .	50521	42366
(oil-hardened)	74371	99401

It will be seen that the numbers re greater for the hardened than for the annealed steel, and m h greater for steel than for wrought iron. Also the numbers the second column are all greater than those in the third, except in the case of the hardened steel.

Retentiveness and Coercive Force Defined.

The descending and second ascending curve show that the induction is not zero when the magnetizing force is zero, but has a value OB. In the descending curve (2) a neg tive magnetizing force OA is required to annul the induction, and in the second ascending curve a positive magnetizing force equal in value to the former is again necessary to reduce the induction to zero. Dr. Hopkinson calls OB (the induction which remains after gradual reduction of the magnetizing force to zero from a large value) the "retentiveness," and the magnetizing force OA, required to annul the induction, the "coercive force" of the material.

The curve marked residual induction was obtained by applying and removing the magnetizing forces represented by the abscissæ, and measuring the induction at each removal. It is supposed by Ewing, chiefly on the ground of the smallness of the residual magnetism, that the condition of endlessness was not perfectly attained. In this case the curves could be corrected by shearing them to the left, that is by measuring the magnetizing forces from a line inclined to the axis of OY, as described above (p. 724).

Experiment on Iron Ring.

The following are results of an experiment of Ewing's* on an iron ring, with the data used in calculating the magnetizing force and induction from the observations. The magnetizing force was increased by steps from a zero value up to 9·14 C.G.S., diminished by steps, and again applied in the same way. The ballistic throw for each step was observed.

Data of Apparatus for Ballistic Experiments.

DATA OF APPARATUS.

Diam. of wire ·248 cm.
Mean circumference of ring 31 4 cms.
Area of each turn of earth inductor 1216 sq. cms.
No. of turns in earth inductor 10.
Earth's vertical force ·34 C.G.S.
Earth inductor reading 42 9.

No of turns in magnetizing coil 474.
No. of turns in induction coil 167.
Deflection of battery galvanometer with 3 Daniell's cells and 6·85 ohms resistance 362 divs.

Phil. Trans. R.S. 1885, or *Magnetic Induction in Iron*, &c. p. 70.

Reduction of Observations.

Hence the current through the magnetizing coil was, taking the electromotive force of a Daniell's cell as 1·1 volt,

$$\frac{3 \times 1\cdot1 \times 10^8}{6\cdot85 \times 10^9} = \frac{3\cdot3}{68\cdot5}, \text{ C.G.S.}$$

Also the number n of turns on the ring per unit of length of its circumference was 474/31·4. Thus the magnetizing force per scale division of the magnetometer was

$$\mathbf{H}_0 = 4\pi \frac{474}{31\cdot4} \times \frac{3\cdot3}{68\cdot5} \frac{1}{362} = \cdot02525.$$

The whole area A' swept over by the earth inductor was $2 \times 1216 \times 10$ in sq. cms., and the area of section of the wire was $\pi \times \cdot124^2$, so that the effective area of the induction coil through which the induction in the iron passed was $167 \times \pi \times \cdot124^2$ sq. cms. Thus by (11) above the induction for one division deflection of the ballistic galvanometer was

$$\mathbf{B}_0 = \frac{2 \times 1216 \times 10 \times \cdot34}{167 \times \pi \times \cdot124^2 \times 42\cdot9} = 23\cdot89.$$

Defect of Ballistic Method.

The table on the next page gives the observed results, and the quantities deduced from them. The fourth column contains for each throw of the galvanometer the sum of that throw and all the throws that precede it. The induction then existing in the iron was taken as proportional to that sum, and was calculated by the formula just given. Any gradual change taking place in the iron between the successive increments of **H** produced no effect, and therefore does not appear in the account. The full induction in the case at any rate of thick rods may be, as we shall see below, very considerably in excess of that calculated from the transient current deflections, in consequence of a gradual creeping of the magnetization in the direction of the relatively much larger change which takes place when the applied magnetizing force is altered.

TABLE OF RESULTS.

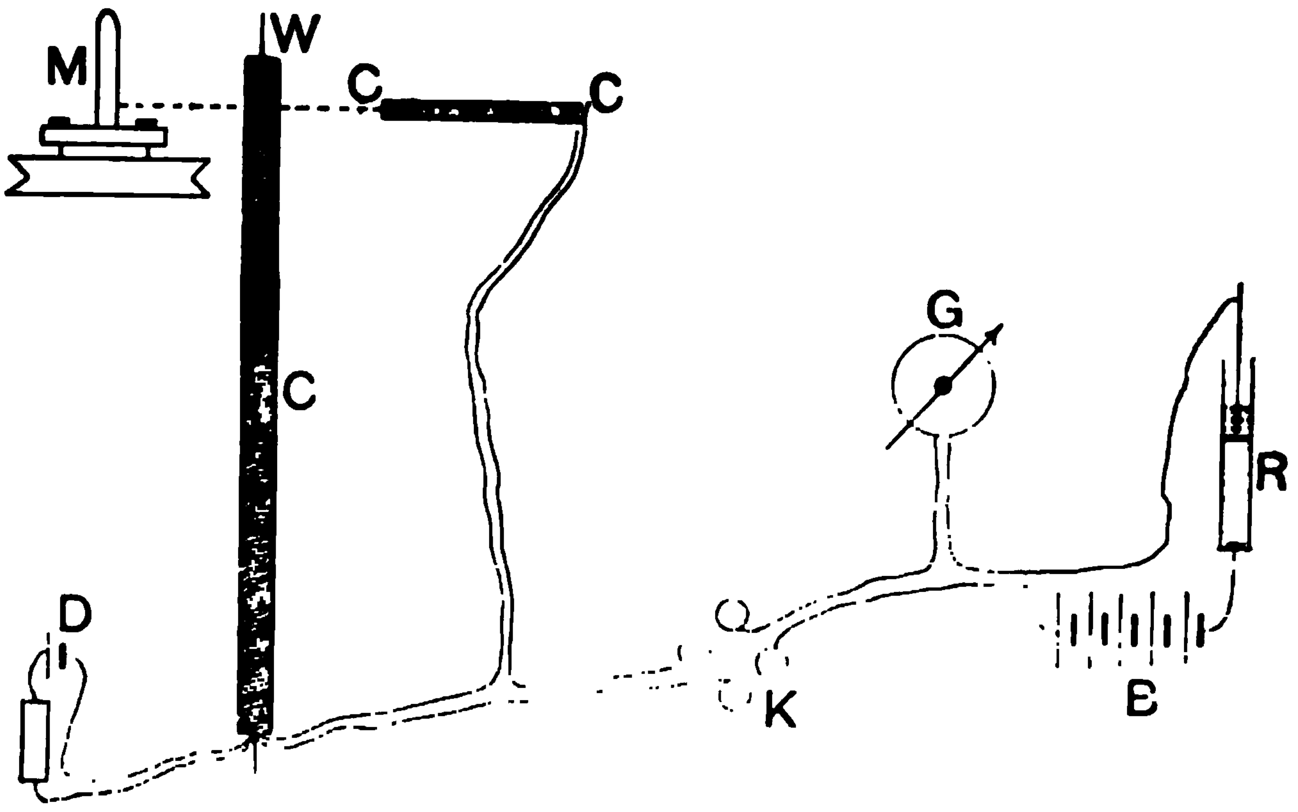
Note. μ, I, and κ have been calculated from **B** and **H** by the equations $\mu = \mathbf{B}/\mathbf{H}$, $\kappa = (\mu - 1)/4\pi$, $\mathbf{I} = \kappa\mathbf{H}$.

Final Results of Series of Ballistic Experiments.

Ball Galv Reading.	H	Ballistic Throw	Sum of Throws.	B	I	κ
5·3	0·13	1·1	1·1	26	2·1	
10·2	0·26	1·1	2·2	53	4·2	
12·0	0·30	0·5	2·7	65	5·1	
16·0	0·40	0·8	3·5	84	6·7	
21	0·53	1	4·5	107	8·6	
28	0·71	2·1	6·6	158	12·5	
37	0·93	2·9	9·5	227	18·0	
52	1·31	3·9	13·4	320	25·4	19
67	1·69	9·2	22·6	540	42·9	25
75	1·89	6·9	29·5	705	56·0	30
110	2·78	77·5	107·0	2560	203	73
133	3·36	78·7	185·7	4440	353	105
159	4·01	82	267·7	6400	509	127
196	4·95	91·5	359·2	8580	683	138
232	5·86	57	416·2	9940	791	135
285	7·20	57	473·2	11300	899	125
321	8·10	23·5	496·2	11870	944	116
362	9·4	24	520·7	12440	989	108
310	7·83	−4·4	516·3	12330	981	
246	6·21	−6·7	509·6	12170	968	
188	4·75	−7·1	502·5	12000	955	
107	2·70	−14·0	488·5	11670	929	
0	0	−33·2	455·3	10880	866	
110	2·78	15	470·3	11240	894	
196	4·95	14·2	484·5	11570	921	
246	6·21	11·9	496·1	11860	943	
317	8·00	14·5	510·9	12170	971	
362	9·14	10	520·9	12440	990	

Disadvantage of Ballistic Method.

As just stated, the ballistic method from its very nature, can take no account of slow changes of magnetization, such as, in the case of rods of any considerable thickness, are found to go on for some time after the magnetizing force has been changed. For this reason the magnetometer method is to be preferred in many cases, if it is possible to apply it. Of course for anchor rings and specimens set in yokes of soft iron so as to give approximate endlessness the ballistic method is the only one available



M. Magnetometer.
C. Magnetizing coil.
W Wire.
CC. Compensating coil.
B. Battery.
K. Reversing key.
G. Current galvanometer.
D. Cell for compensating earth's vertical force.
R. Rheostat.

FIG. 159.

Magnetometric Method.

Fig. 159 shows the general arrangement of apparatus for the magnetometric method. The magnetometer and scale, &c., described in Chap. III. above, are here made use of, and for details of setting up the instruments, taking readings, &c., the reader is referred to that Chapter. Supposing the directive force on the needle to be the earth's horizontal magnetic force H, the specimen with its magnetizing coil is arranged so as to give a horizontal magnetic field at the needle at right angles to

that of the earth. For this purpose the bar may be fixed in the east and west direction, in a horizontal line through the centre of the needle, or preferably in a vertical position in the vertical plane passing through the centre of the needle, at right angles to the magnetic meridian. This position is the more convenient as it enables, if necessary, stress of measured amount to be applied to the bar or wire; it also renders the action of the farther end of the bar or rod relatively unimportant. It involves however the application to the rod of the earth's vertical magnetic force, which must be taken carefully into account, or carefully compensated.

Compensation of Magnetic Action of Magnetizing Coil on Needle.

The coil itself produces, when a current is flowing in it, a direct effect on the needle, but this can be neutralized by means of a compensating coil, in which the same current flows, so placed that when the needle is acted on by the current in the two coils it is not deflected. The compensating coil may be a circle of one or more turns of wire placed like a tangent galvanometer coil in the plane of the magnetic meridian with the centre of the needle at a point on its axis, or it may be a solenoid placed horizontally at right angles to the magnetic meridian, either in a line through the centre of the needle, or in the side-on position of Fig. 13', p. 74 above.

Prevention Direct Action of Connecting Wires.

The connections between the coils and with the battery &c. must all be made with well insulated wire, closely twisted together to prevent direct action of the connections on the needle, and for this reason no open loop must be permitted to exist on the conducting wires near enough the needle to affect it. For example, if the current goes in at one end of the magnetizing solenoid, the wire should be led close along the solenoid from one end to the other, then the two wires twisted together for a sufficient distance from the magnetometer.

Determination of Directive Force at Needle

We shall suppose that H, the horizontal magnetic force at the needle, is known. It can in any case be obtained approximately enough by experiment either in the manner described above, or if H is known at another place it may be obtained at the place of experiment by observing the period of free vibration of a needle at the two places. Thus if H be the required value, and H' the known value at the other place, T, T', the corresponding periods

$$H = \frac{T'^2}{T^2} H' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Or a ring of one or more turns of wire may be set up in the magnetic meridian with the needle at its centre, or on its axis.

and a current, the absolute value of which is known by electrolysis, or from measurement by an absolute current meter, sent through it, and the magnetometer deflection θ determined. If G be the galvanometer constant of the coil, and γ the current, we have $\gamma = H \tan\theta/G$, or

$$H = \frac{G\gamma}{\tan\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

If a compensating circular coil be used it may be employed for this purpose also. The value of G may be obtained by comparing the expression $G\gamma\cos\theta$ for the couple on the needle produced by the current with that on the right of (13) of Chap. VI, above.

Elimination of Earth's Vertical Force. The vertical component of the earth's magnetic force may be allowed for; or it may be permanently eliminated by winding a layer of wire on the solenoid, and connecting it to a cell adjusted by resistance in its circuit so as just to produce a field equal and opposite to that of the earth. The strength of the current necessary for this purpose may be adjusted by placing a piece of soft iron wire within the coil, hanging a scale pan to its lower end, and applying and removing a number of times in succession a weight of 7 or 8 lbs., thus subjecting the wire to a series of alternate elongations and shortenings, until the wire shows no magnetization. This process was followed by T. Gray and the writer, when assisting to carry out Lord Kelvin's researches on the Effects of Stress on the Magnetization of Iron.*

Another process, followed by Ewing, differs from this only in substituting a succession of reversals of magnetism produced by a series of currents alternately in opposite directions, and each slightly weaker than the preceding. Thus the iron is at length completely demagnetized if the earth's force is annulled by the current. If the annulment is not complete a certain amount of magnetization in one direction or the other will always be left.

Theory of Magnetometric Method. Supposing now the effect of the coil, and the vertical magnetic force of the earth all carefully compensated, and the bar placed in position with its upper end near, and due magnetic east or west of, the needle, the effect on the needle will be mainly due to the upper end of the bar. The bar is moved up or down in the magnetizing solenoid until for a given current the greatest deflection of the needle is produced, and it is then secured in position. Thus denoting the length of the bar or

* *Phil. Trans. R.S.* 1878.

wire from the level of the needle to the lower end by l, the distance of the bar from the needle by x, the intensity of magnetization (that is, the magnetic moment per unit of volume) by $\mathbf{I}$, and the cross-section of the bar by a, the total horizontal force at the needle produced by the bar is

Calculation of Intensity of Magnetization.

$$\mathbf{I}a\left\{\frac{1}{x^2} - \frac{x}{(x^2 + l^2)^{\frac{3}{2}}}\right\} = \frac{\mathbf{I}a}{x^2}\left\{1 - \frac{x^3}{(x^2 + l^2)^{\frac{3}{2}}}\right\}$$

Thus for equilibrium of the needle

$$\frac{\mathbf{I}a}{x^2}\left\{1 - \frac{x^3}{(x^2 + l^2)^{\frac{3}{2}}}\right\}\cos\theta = H\sin\theta$$

or

$$\mathbf{I} = \frac{x^2}{a}\,\frac{H\tan\theta}{1 - \dfrac{x^3}{(x^2 + l^2)^{\frac{3}{2}}}} \quad . \quad . \quad . \quad . \quad . \quad (33)$$

If l be great in comparison with x we may take it that

$$\mathbf{I} = \frac{x^2}{a}\mathbf{H}\tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

From $\mathbf{I}$ of course $\mathbf{B}$ can at once be found by the equation

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I}.$$

Of course for $\mathbf{H}$ must be taken the magnetizing force applied by the coil, together with if necessary a correction for the effect of the ends (see p. 722 above).

Determination of Permeability. Effects of Stress on the Magnetization 1. Of Iron.

The permeability of a specimen of iron is obtained at once by calculating the ratio $\mathbf{B}/\mathbf{H}$. Thus it is numerically equal to the tangent of the inclination of the curve of induction at each point to the axis of abscissæ. It is clear, from the example given, that for low magnetizing forces the permeability is small (if the specimen is not subjected to vibration); then as the magnetizing force is increased the permeability at first rapidly increases, then more slowly, then diminishes, and finally approaches zero for very high magnetizing forces.

2. Of Steel, Nickel and Cobalt.

That the permeability of steel, nickel, and cobalt is diminished by the application of elongating stress, and increased by its withdrawal, and is therefore increased by compressing stress, was observed in Lord Kelvin's experiments above referred to. The permeability of soft iron it was found was increased by the

Lord Kelvin's Experiments. Villari "Critical Value" of Magnetizing Force.

application of elongating stress so long as the magnetizing force was below a certain value. At first the effect of the stress on the magnetization increased with the magnetizing force to a maximum, then fell off, and finally became zero with a certain magnetizing force, depending on the amount of stress, thereafter changing sign. This phenomenon had previously been observed for soft iron by Villari, and hence the magnetizing force for which the effect of stress was zero was called the *Villari critical value.*

Transverse Stress.

Pressure applied to the interior surface of an iron tube (an *iron* gun barrel), in other words stretching force round every cross-section of the tube, produced the opposite effect, that is diminished the magnetization while the magnetizing force was under a certain value and increased it when the magnetizing force was above that value.

Effects of Vibration and Mechanical Disturbance.

In connection with effects of stress it may be noticed here that mechanical vibration has a notable effect in aiding magnetic changes in soft iron and to a less degree in steel, nickel, &c. A succession of applications and removals of stress were found in Lord Kelvin's experiments to be very effectual in bringing a wire or bar of magnetic material to a permanent magnetic state. By this process the magnetization was in general increased when the specimen was under the influence of a given magnetic force. Stripping gently a wire of soft iron through the fingers, for example, was found greatly to aid its magnetization or demagnetization.

By aid of tapping, Ewing was able almost completely to obliterate the concavity at the beginning of the curve of magnetization for a soft iron wire, and to reach near the beginning of the curve a permeability (**B**/**H**) of nearly 80000.

Magnetization in Intense Field.

Magnetization with powerful magnetic forces has been studied by Ewing and Low and by Du Bois. The former used what they called the isthmus method, in which a short cylindrical piece of the material forming a bobbin wrapped round with two induction coils was placed as a "neck" or "isthmus" between two truncated conical pole-pieces, attached to a powerful electromagnet and shaped so as to produce as great an intensity of field as possible. The cones were placed with their axes in line so as to form a complete (or double) cone with the vertex at the centre of the neck. The bobbin had its ends turned so as to fit in a cylindrical space the axes of which passed exactly through the common vertex of the conical pole-pieces at right angles to their axis. A brass holder, fitting in this space, enabled the bobbin to be turned round when required through 180°, so as to be reversed in the field. One of the

Isthmus Method.

induction coils consisted of one or two layers and was wrapped round the neck, the other was separated from the former by an annular space which enabled the intensity of the field close to the neck to be estimated from the difference between the inductions through the two coils.

Theory of Action of Conical Pole-pieces.

To find the action of the conical pole-faces consider a ring of either pole-face at distance x from the vertex, of radius r, and breadth dr parallel to the radius. The magnetic distribution in the ring is $2\pi\mathbf{I}r dr$. The force which this produces parallel to the axis is $2\pi\mathbf{I}xr dr/(x^2+r^2)^{\frac{3}{2}}$. If we call this dF, we know that it is a maximum when $\partial(dF)/\partial x$ is zero, r being taken constant since the diameter of the pole-pieces is fixed. The condition is $2x^2 = r^2$, or if θ be the semi-vertical angle of the cone, $\tan\theta = \sqrt{2}$, and $\theta = 54° 44'$.

The whole force produced can be calculated by integrating dF over the pole-faces. Let a be the radius of the neck, b the maximum radius of either pole-face, we have supposing each pole magnetized to constant intensity $\mathbf{I}$

$$F = 4\pi\mathbf{I}\int_a^b \frac{xr\,dr}{(x^2+r^2)^{\frac{3}{2}}} = \frac{4\pi\mathbf{I}}{\sqrt{3}}\int_a^b \frac{r\,dr}{x^2+r^2}$$

$$= \frac{8\pi\mathbf{I}}{3\sqrt{3}}\int_a^b \frac{dr}{r} = \frac{8\pi\mathbf{I}}{3\sqrt{3}}\log_e\frac{b}{a} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

since $x^2 = r^2/2$.

Taking $\mathbf{I}$ as 1700, the saturation value nearly for wrought iron, we get

$$F = 11{\cdot}13\,\mathbf{I}\log_{10}\frac{b}{a} = 18920\log_{10}\frac{b}{a} \quad . \quad . \quad . \quad (36)$$

Results of Experiments by Isthmus Method.

Experiments were made with this arrangement and also with cones of semi-vertical angle $\theta = \tan^{-1}\sqrt{\frac{2}{3}}$ or $\theta = 39° 14'$ which were chosen as giving the most uniform field about the axis on the supposition of saturation of the pole-pieces everywhere.

In the first case with a neck of ·266 mm. diameter the following were the results:—

H	B	I	μ
24,500	45,350	1,660	1·85

Results of Experiments.

With the cones of smaller diameter the following is one of the sets of results obtained:—

Swedish Iron "L Lancash." Brand.

H	*B*	*I*	*μ*
1,490	22,650	1,680	15·20
3,600	24,650	1,680	6·85
6,070	27,130	1,680	4·47
3,600	30,270	1,720	3·52
18,310	38,960	1,640	2·13
19,450	40,820	1,700	2·10
19,880	41,140	1,700	2·07

Fine Swedish Iron "L" Brand.

5,310	25,670	1,620	4·83
17,680	38,080	1,620	2·15
19,240	39,540	1,620	2·06

Magnetization in Intense Fields according to Ampère's Theory.

These experiments were instituted to test whether the intensity of magnetization really attained a maximum and thereafter diminished as the magnetizing force was pushed to higher and higher values. According to the Amperean theory of magnetization the magnetic molecules are supposed to be small conducting circuits carrying currents. After these have as far as possible been turned into a common direction by the action of the magnetizing force, further increase of the field-intensity ought to have the effect of diminishing the intensity of magnetization by inductive diminution of the molecular currents. Thus, as Maxwell has remarked,* "If it should ever be experimentally proved that the temporary magnetization of any substance first increases and then diminishes as the magnetizing force is continually increased the evidence of the existence of these molecular currents would, I think, be raised almost to the rank of a demonstration."

No such diminution of intensity of magnetization is shown by the experiments quoted, or by others by the same method made on steel, nickel, and cobalt; on the contrary nearly full intensity of magnetization is reached with comparatively low

* *El. and Mag.* vol. ii. p. 436 (2nd edition).

magnetizing forces, and a tenfold increase after that gives practically the same value for **I**.

Experiments of Du Bois.

These conclusions agree with those of Du Bois* made by an ingenious optical method, in which the elliptic polarization, produced by the reflection of plane polarized light from the polished pole of an electromagnet, was used to measure the intensity of magnetization.

Optical Method for Intense Fields

Du Bois first determined† by experiments on plane polarized light incident on small reflecting planes ground in various positions on ovoids (prolate ellipsoids of revolution) of different materials, comprising iron, steel, nickel, cobalt, and magnetic (magnetite oxide of iron), the general law which Kerr's phenomenon followed.

Reflection of Plane Polarized Light at Magnetized Surface.

The nickel contained traces of iron and copper, the cobalt 93·1 per cent. cobalt, 5·8 per cent. nickel, ·8 per cent. iron and ·2 per cent. copper. The ovoids were magnetized in a coil 30 cms. long, 4 cms. in inner and 12 cms. in outer diameter, and composed of twelve layers of 90 turns each of double cotton-covered copper wire, further insulated with shellac. The wire was ½ cm. in diameter and the resistance of the coil when cold was ·9 ohm. It was possible to surround the coil with an ice jacket to keep down its temperature.

The ovoids themselves could be maintained at 0° or 100° by an ice or steam jacket inside the magnetizing solenoid.

In the winding of the coil small tubes ·7 cm. in diameter were fixed to allow the polished planes in the various positions on the ovoids to be viewed from the outside of the coil. This did not produce any sensible alteration in the uniformity of the field.

The intensity of the field, **H**, was determined in the usual manner from the strength of the current, which was measured by means of an amperemeter. A demagnetizing force of ·52 **I** was allowed for in the reckoning of **H**. (See p. 723 above.)

Determination of "Kerr's Constant."

In all the experiments the light was incident normally on the reflecting surface, and the quantity measured was the angle between the major axis of the vibrational ellipse after reflection, and the direction of vibration in the unreflected ray, or what is commonly but rather incorrectly named the rotation of the plane of polarization produced by the reflection. It was found that if K be a constant (called Kerr's constant by Du Bois), and θ the angle between the normal to the surface and the direction

* *Phil. Mag.* April, 1890.

† *Phil. Mag.* March, 1890, or *Wied. Ann.* 39 (1890).

of magnetization, ϵ the angle observed was given by the relation

$$\epsilon = K\mathbf{I}\cos\theta = K\mathbf{I}_n \quad . \quad . \quad . \quad . \quad . \quad (37)$$

where $\mathbf{I}_n$ denotes the component intensity of magnetization at right angles to the reflecting surface. The following are numerical results:

	Cobalt.	Nickel.	Iron.	Magnetite.
ϵ	$-20'97$	$-7'\cdot25$	$-22'\cdot99$	
I	1060	453	1669	
K	$-\cdot0198$	$-\cdot0160$	$-\cdot0138$	$+\cdot012$

The minus signs indicate that the direction of rotation was opposite to that in which a right-handed screw would have had to be turned so as to advance along] the direction of magnetization. The value of K was found to be hardly sensibly affected by change of temperature.

Use of Kerr's Constant in Optical Method.

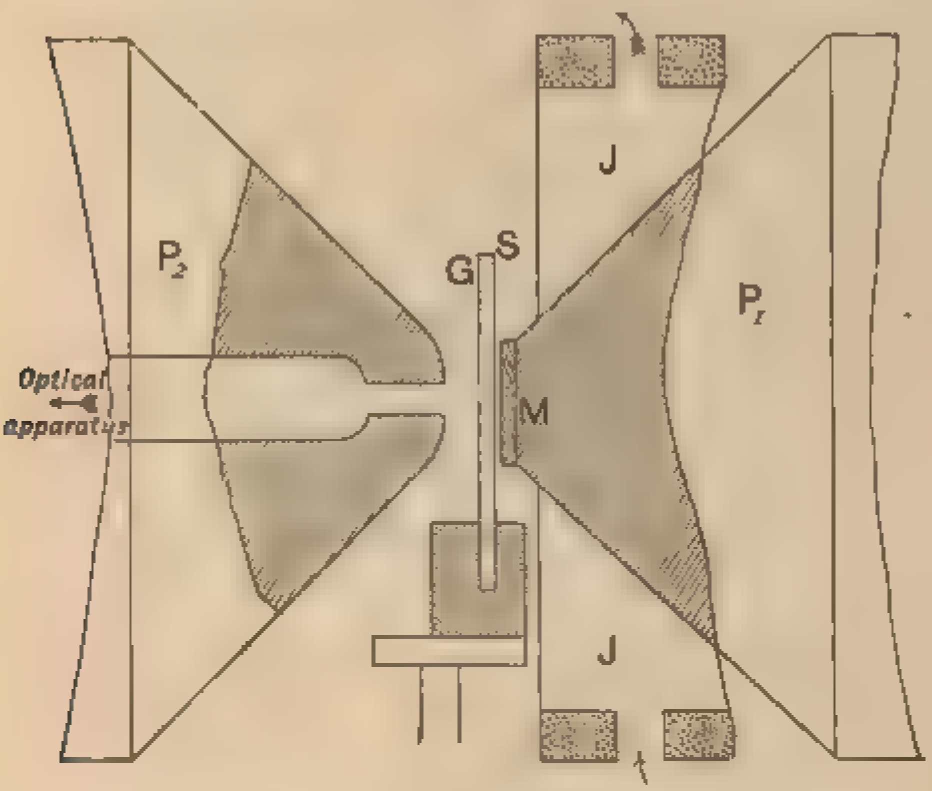
To obtain results for very intense fields small disks were turned out of the materials already specified, and polished on one face. Each of these when examined was fixed with the polished face outwards to the conical point P_1 (Fig. 160) of one pole of a Ruhmkorff electro-magnet. A beam of plane polarized light was made normally incident on the mirror, through the axial perforation in the opposite pole-piece, and was examined before and after reflection by optical apparatus placed as indicated in the figure. The pole point and the mirror specimen, M, could be kept nearly at 100° by means of a steam jacket J, J.

Method Determining H

The intensity of magnetization, I, was calculated from the results of observation by means of the values of Kerr's constant previously found. The field intensity at the mirror was found by placing a glass plate, G, silvered on the side, S, close in front of the mirror, and observing the magnetic rota-

tion due to the double passage of the light through it. The glass plate had been standardized by comparison with bisulphide of carbon, from which Verdet's constant (see p. 765 below) is accurately known. The magnetic rotation experienced by the ray in its passage in the air from and to the optical apparatus was determined and allowed for.

The value of the field intensity **H** outside the specimen was of course by the continuity of magnetic induction precisely



M Mirror specimen.
*P*₁ Conical pole-piece.
*P*₂ Perforated pole-piece.
G Plate of glass silvered at *S*
J, *J* Jacket for steam or ice

FIG. 160.

equal to the induction inside the specimen. Thus the magnetizing force **H** within the specimen was found by the relation

$$\mathbf{B} = \mathbf{H}' = \mathbf{H} + 4\pi\mathbf{I},$$

or

$$\mathbf{H} = \mathbf{H}' - 4\pi\mathbf{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Du Bois' results for nickel and cobalt are given in the following table, and illustrated for iron and steel in Fig. 161. The ordinates in the curves are values of the magnetization intensity per gramme of the material, and require simply multiplication

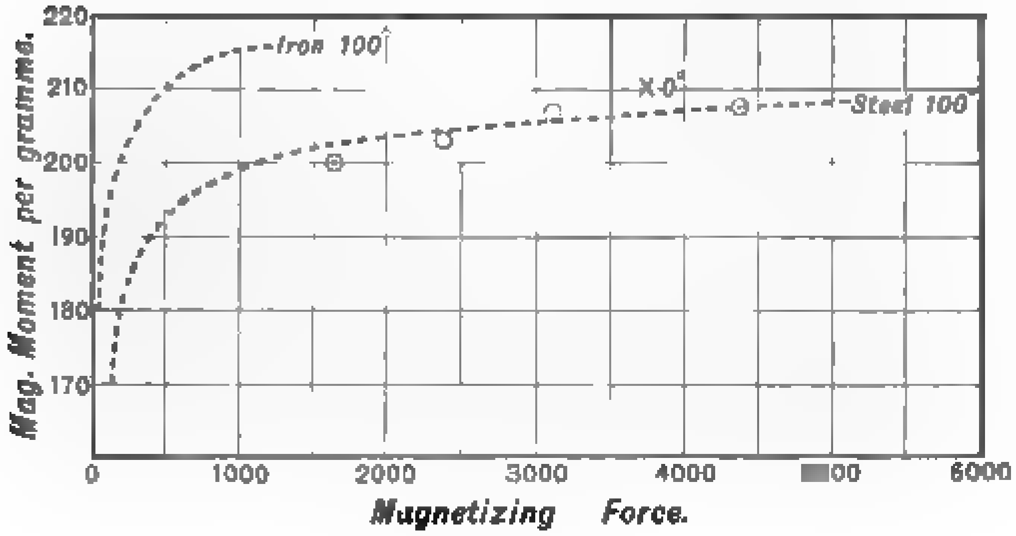


Fig. 161.

by the density to give **I**. The abscissæ are values of **H**, the magnetizing force within the specimen.

Cobalt. Temp. 100° C.			Nickel. Temp. 100° C.		
B	ε	**I**	**B**	ε	**I**
16750	−22′·45	1134	9920	−8′·29	518
19550	−23′·24	1174	12850	−8′·36	522
21710	−23′·38	1181	16250	−8′·43	527
23330	−23′·60	1192	19290	−8′·40	525

Magnetization with Small Magnetic Forces. Experiments have been made by Baur, Lord Rayleigh, and Ewing on the magnetization of iron by small forces. Baur found by ballistic experiments on a ring of soft iron that for low magnetizing forces the permeability and susceptibility of

iron are nearly constant. His results are expressed by the equations

$$\left.\begin{aligned} \kappa &= 14{\cdot}5 + 110\,\mathbf{H} \\ \mu &= 183 + 1382\,\mathbf{H} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (39)$$

which hold for values of **H** from ·0158 to ·384.

These give

$$\left.\begin{aligned} \mathbf{I} &= 14{\cdot}5\,\mathbf{H} + 110\,\mathbf{H}^2 \\ \mathbf{B} &= 182\,\mathbf{H} + 1382\,\mathbf{H}^2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (40)$$

Baur's Experiments

a parabolic relation which holds for many pairs of mutually varying physical quantities when the limits are narrow enough.

Lord Rayleigh's Experiments.

Lord Rayleigh arranged an unannealed iron wire for test by the magnetometric method, and compensated by means of a coil the total action on the needle when the magnetizing force was ·04 C.G.S. It was found then that when the magnetizing force was brought down gradually to ·00004 C.G.S. the compensation remained perfect. This proved that the magnetization was proportional to the magnetizing force throughout the whole range of variation. For magnetizing forces above ·04 the proportionality did not hold, and up to the value 1·2 C.G.S. for **H** the results were expressed by

$$\left.\begin{aligned} \kappa &= 6{\cdot}4 + 5{\cdot}1\,\mathbf{H} \\ \mu &= 81 + 64\,\mathbf{H} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (41)$$

from which **I** and **B** can be found as before.

Similar results were obtained for nickel and steel.

Magnetic Viscosity in Annealed Iron.

With unannealed iron or steel Lord Rayleigh found that if balance was obtained with the compensating coil at the moment of closing the battery current no disturbance of the compensation took place afterwards. This showed that these substances took their complete magnetization at once. When the iron was soft however an apparent magnetic viscosity displayed itself. When the instantaneous effect was reduced to zero the needles, after the putting down of the key, drifted round in the direction showing an increase of magnetization.

Magnetic Cycle due to Viscosity.

This result was studied by Ewing in some further experiments, which showed that a piece of iron could be put through a complete cycle by first applying the current and then after a minute removing it. A force of ·044 C.G.S. applied gave an instantaneous value of **I** = ·44, after five seconds **I** had become ·58, and after sixty seconds ·67. Removal of the force gave at once a diminution of **I** by ·44, after five seconds the remaining

·23 had fallen to ·09, and after sixty seconds to zero. Thus the falling off of the magnetization followed the same law as its increase. If the variations of magnetic state of the iron during the cycle were represented graphically by a curve of intensity of magnetization **I** for different values of **H**, the area, $\int \mathbf{H}d\mathbf{I}$, of the closed loop would represent the energy dissipated.

Dissipation of Energy.

Effects of Temperature on Magnetic Susceptibility.

The effects of varying temperature on magnetization are very remarkable, but we have not space to do more here than allude to them. In wrought iron and steel rise of temperature generally increases the magnetic susceptibility for small magnetizing forces, and diminishes it for high forces. When however the temperature is raised nearly to that of redness zero susceptibility sets in rather suddenly, and at the same temperature whatever the magnetizing force. This temperature varies in ordinary iron and steel with the nature of the specimen. According to Hopkinson's experiments* it is the temperature at which cooling iron, after it has become almost dark, suddenly reglows.

Behaviour of Nickel-Steel at Different Temperatures.

A kind of steel containing 25 per cent. of nickel was found by Hopkinson to be unmagnetizable at ordinary temperatures, but to become magnetizable at a temperature a little below the freezing point, and then to remain so up to 580° *C.* It then became unmagnetizable, and did not regain susceptibility when cooled to ordinary temperatures.

SECTION III

DISSIPATION OF ENERGY IN CYCLES OF MAGNETIZATION

MOLECULAR THEORY OF MAGNETISM

Energy in Magnetic Cycles.

It is shown at p. 213 above that the energy spent otherwise than in increasing the electrokinetic energy in a step from P to Q (Fig. 162) on the curve of induction is

$$\frac{1}{4\pi}(\text{area } PQSR - \tfrac{1}{2} \text{ area } NQSRPM).$$

* *Phil. Trans. R.S.* 1889, A. p. 443.

It is interesting to apply this result to an actual curve of magnetization in iron (Fig. 163). For all points on the curve up to a little distance beyond P, the total energy given to the

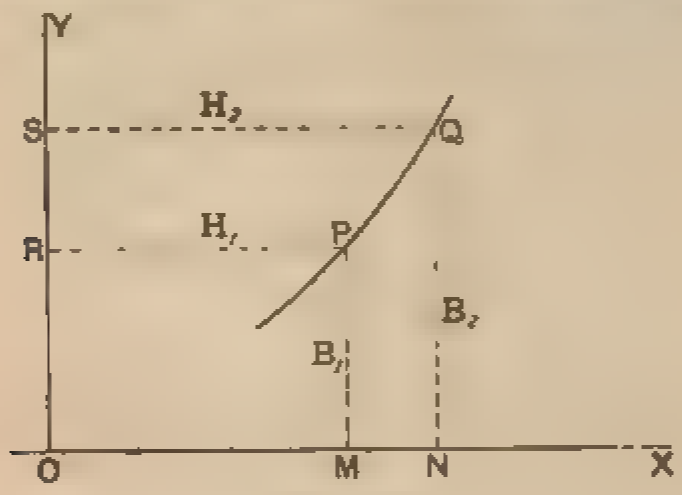


FIG. 162

medium in producing the corresponding magnetization exceeds the electrokinetic energy, but for all points further from the origin the electrokinetic energy exceeds, and for points on the upper flat part of the curve very greatly exceeds, the energy

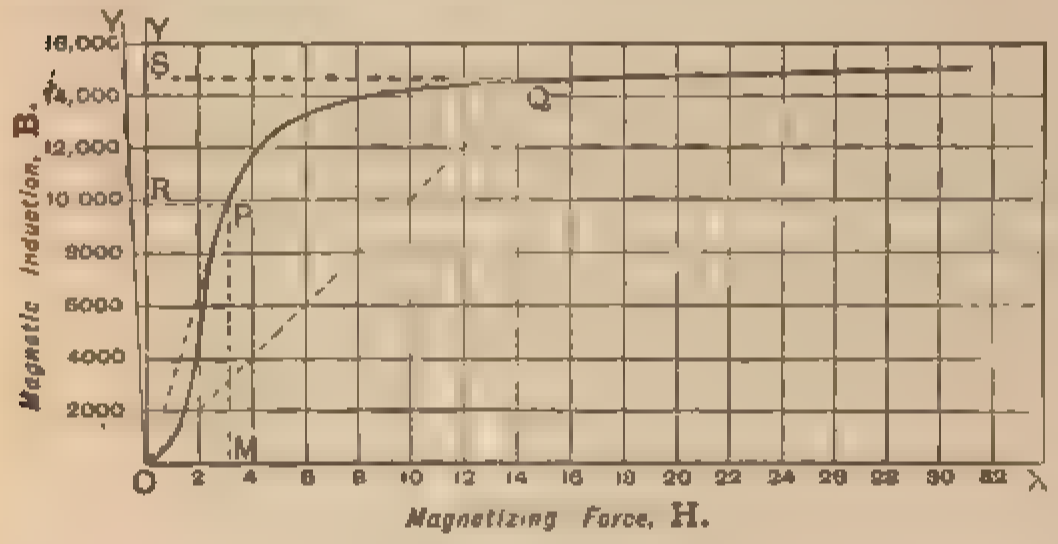


FIG. 163

upper flat part of the curve very greatly exceeds, the energy given out to the medium by the battery. [The point P at which OP is a tangent to the curve marks the point at which

the energy given out to the medium and the electrokinetic energy are increasing at the same rate; the former below that point is increasing faster, above that point slower than the latter.]

Energy in Steps of Magnetization.

We are forced to conclude that for every series of magnetization steps from zero up to a certain point energy is given to the medium, and for every series from zero up to any further point the medium furnishes the balance of energy required for the electrokinetic energy. In every small step below P a balance of energy, over and above the electrokinetic energy, is given to the medium; for every small step above P a quantity of energy is taken from the medium to make up the electrokinetic energy.

When however a complete cycle of changes is performed we are able to say definitely that so much energy has been dissipated in the form of heat in the iron. It follows from the above expression (see also p. 212 above) since the second area vanishes that the energy W dissipated is given by the equation

$$W = \frac{1}{4\pi}\int \mathbf{H}d\mathbf{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

the integral being taken round the closed curved formed by the curves of induction for the forward and backward parts of the cycle.

This theorem was given first by Warburg* and afterwards by Ewing.†

Dissipation of Energy in a Magnetic Cycle, or "Hysteresis."

The dissipation arises through a lagging of the changing magnetization of the iron behind the magnetizing force to an extent dependent on the previous history of the iron. This lagging action has been called by Ewing "hysteresis," and the name seems now generally adopted.

Equation (42) may be written

$$W = \int \mathbf{H}d\mathbf{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42')$$

the most convenient form for calculation.

Fig 164 shows a cycle of magnetization for a soft iron ring. The results were of course obtained by ballistic experiments.

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* *Wied. Ann.* xiii. (1881), p. 141.
† *Proc. R. S.* May, 1882.

The smaller loops were formed by diminishing to zero and reapplying the magnetizing forces indicated at the different places.

Fig. 165 gives a cycle for an annealed iron wire of length 400 times the diameter, and Fig. 166 a cycle for a wire of annealed steel. It will be noticed that the curve is of much greater area in the latter case than in the former.

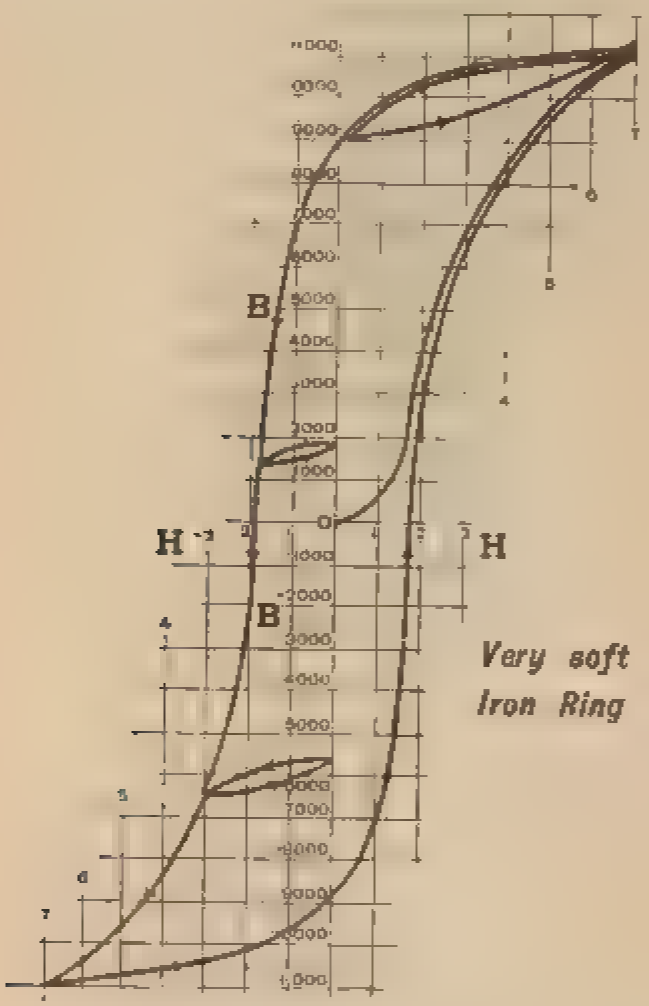

FIG. 164.

The value of $\int \mathbf{H}d\mathbf{I}$ for the double reversal shown in Figs. 164, 165, is about 10,000 ergs per cubic cm. As there are about 7·7 grammes of iron in a cubic cm., this amount of energy,

Estimate of Energy Dissipated in Double Reversal. if it were retained in the iron, would raise it through a difference of temperature

$$\frac{10000 I}{41{\cdot}6 \times 10^6 \times 7{\cdot}7 \times {\cdot}11} = 2{\cdot}84 \times 10^{-4}$$

of 1° *C*, where $41{\cdot}6 \times 10^6$ ergs is taken as Joule's equivalent of

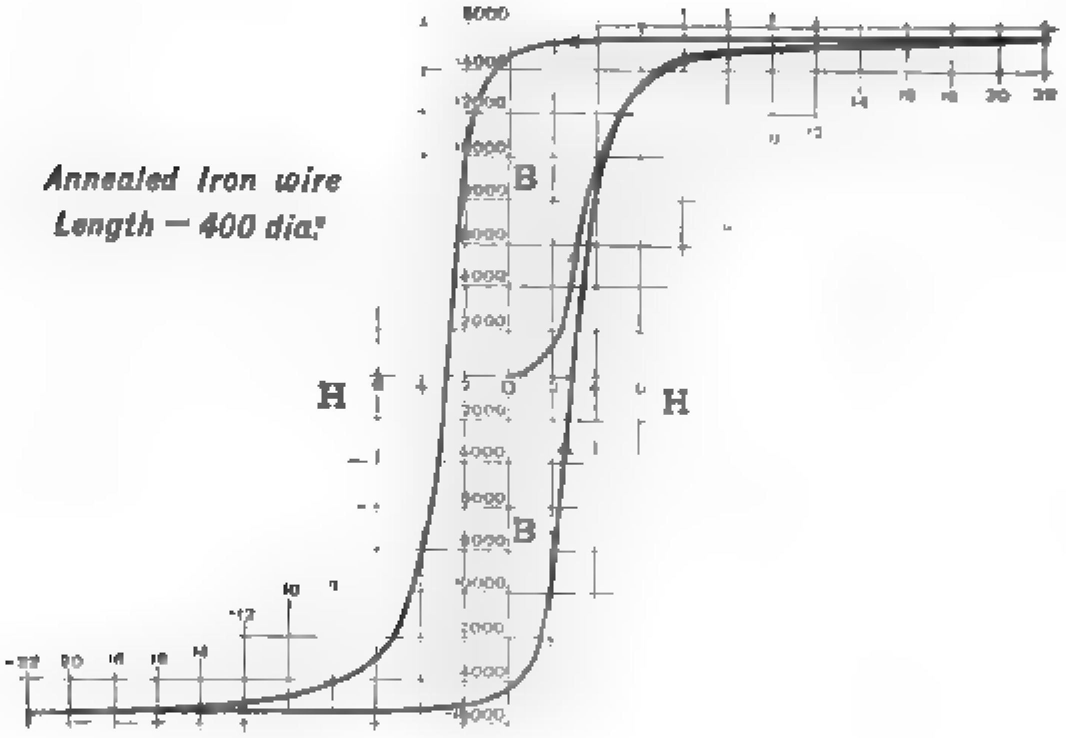

FIG. 165.

heat, and ·11 as the specific heat of iron. In general the rise of temperature for any cycle is

$$2{\cdot}84 \times 10^{-8} \int \mathbf{H}\, d\mathbf{I}.$$

Rate of Dissipation of Energy in Hysteresis. The rate at which work is spent in magnetic hysteresis in a ton of iron, when the number of cycles or double reversals is 100 per second, is according to these figures nearly 18 horse power.

The work spent in hysteresis is much greater in steel than in iron, and is greater for hard than for annealed steel. For

reversals of strong magnetization it is, as we have seen at p. 732 above, approximately 4 × coercive force × intensity of magnetization. The amount varies, according to the table of Dr. Hopkinson's results given above (p. 732), from about 17,000 ergs per cycle for wrought iron to 74,000 ergs per cycle for mild steel hardened with oil, and increases, other things being equal, with the percentage of carbon in the steel.

In chrome steel, oil-hardened, Hopkinson found that 169,000 ergs were dissipated per cycle per cubic cm. of the material,

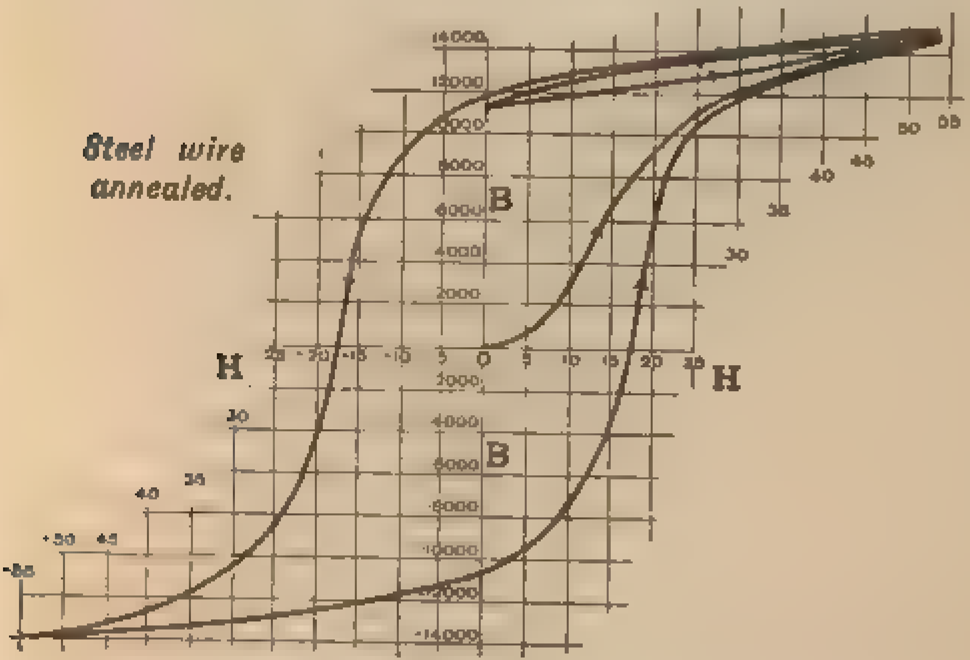

Fig. 166.

and for French tungsten steel, oil-hardened, as much as nearly 217,000 ergs per cubic cm. It was found by Ewing that for small ranges of induction the waste of energy is much less than for larger ranges. High inductions ought therefore to be avoided in alternating dynamos and transformers.

Hysteresis in Cycles of Different Frequency. A considerable amount of attention has recently been directed to the question of the amount of energy dissipated in cycles of magnetization performed at different speeds. Experimental results on this point obtained by Ewing and others are quoted at p. 686 above. Recently the subject has been investigated by

Messrs. J. and B. Hopkinson,* by Messrs. Evershed and Vignoles,† and by Prof. Ayrton and Dr. Sumpner.‡

Hopkinson's Experiments.

Messrs. Hopkinson experimented as follows. A ring was made by winding varnished iron wire in a circular hank, and joining then the ends of the wire. A magnetizing coil was wound round the ring and then joined in series with a non-inductive resistance in the circuit of an alternating machine, as at H and R in Fig. 167. A key L and connections were arranged so that the terminals of the quadrant electrometer indicated in the figure could be connected between C and D or between D and E. A revolving ebonite disk keyed to the axle of the generator connected the electrometer terminal to D once in each revolution, by bringing a stud on its edge into contact for an instant with a steel brush. The disk could be set so that the contact could be made at any stage of the alternation. [P Q was a

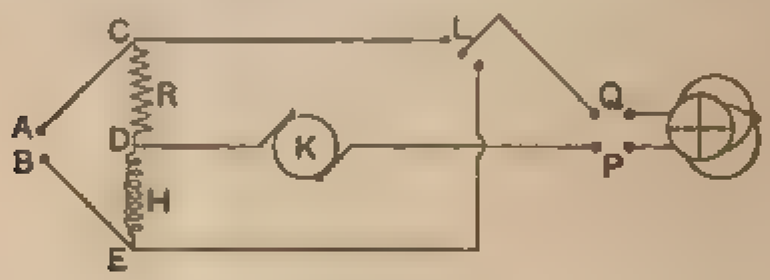

Fig. 167.

reversing key which enabled the readings to be taken in either direction on the electrometer. A condenser was used between P and Q to steady the readings.]

The readings for the points CD were proportional to the current flowing in the circuit (and therefore to the magnetizing force), since there being no inductance in that part of the circuit the difference of potential between C and D and the current were in the same phase. These gave the ordinates by which the curves A in Figs. 168, 169 were plotted with abscissæ proportional to time.

The readings however for the points D, E, were proportional to $dN/dt + R_1\gamma$, if N denote the whole induction through the magnetizing coil H, and R_1 the resistance of the coil. Hence

* *Electrician*, Sept. 9, 1892.
† *Electrician*, Sept. 30 and Oct. 7, 1892.
‡ *Electrician*, Oct. 7, 1892.

by diminishing the readings by an amount proportional to $R_1 \gamma$, numbers proportional to dN/dt were obtained. Of course the values of $R_1 \gamma$, could be subtracted at once by plotting the actual readings in a curve to the same axes as A, and then shortening the ordinates by lengths equal to the corresponding ordinates of A each multiplied by R_1/R. The values of dN/dt were plotted in a second curve B, alongside A.

The area between any ordinate of B, the axis of abscissæ, and the portion of the curve lying between the point of crossing the axis of abscissæ and the ordinate was therefore proportional to the total induction through the iron at the instant corresponding

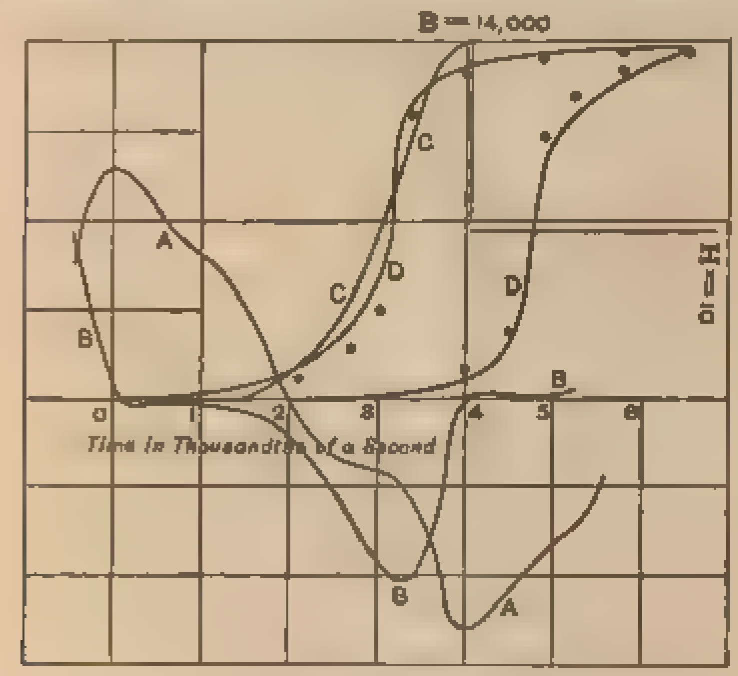

FIG. 168.

to the ordinate. This was plotted as a third curve C. From this the hysteresis cycle was obtained, and was plotted on the diagram.

[The total period was 8/1000 of a second represented by the total length of each Figure along the line of abscissæ. The reader will have no difficulty in completing a full period of the curves A, B. It will be noticed that neither is even approximately a simple curve of sines; also that, as it ought, the curve B cuts the axis of abscissæ at the maxima of A.]

Figs. 168, 169 give the results for soft iron and hard steel at the frequency 125, and between the maximum positive and negative

inductions shown. It will be noticed that the extreme values of the induction agree in the two cases, but that the slow curve falls within the other except for a short distance after the maximum positive or negative induction has been reached and the return curve begun.

The area of the curve is thus greater for the rapid than for the slow cycles, but there is no sign of magnetic viscosity rendering the extreme inductions reached different in the two cases.

Evershed and Vignoles' Experiments.

Messrs. Evershed and Vignoles however have found that when a cycle is performed very slowly there is a perceptible "creeping" of the magnetism in the steep part of the curve which is sufficient to account for the effect observed by Messrs. Hopkinson. In an elaborate series of experiments (see

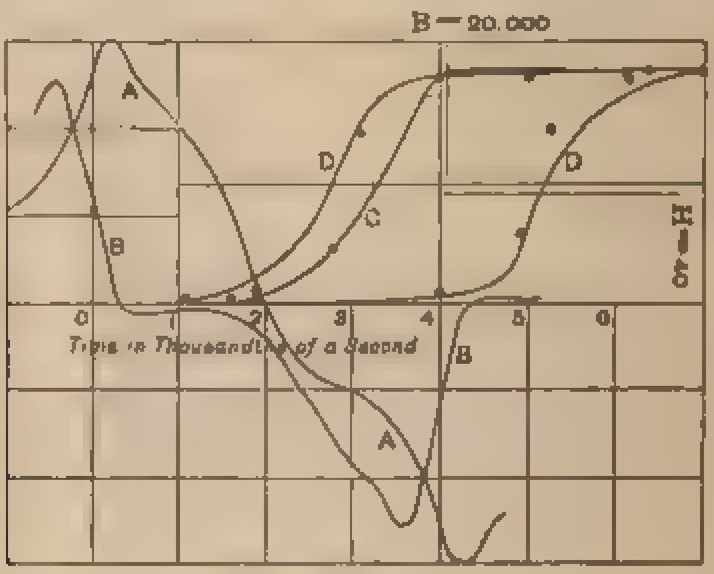

Fig. 169.

loc cit.) they measured the rise of temperature produced by putting the iron core of a transformer through a series of rapid cycles, and having determined by direct experiment the thermal capacity of the transformer, thence deduced the amount of heat generated in the iron in consequence of hysteresis. This was then compared with the heat which would have been produced if the iron had been put through its cycle in about two seconds, and which was estimated from slow cycle observations, together with the (much smaller) calculated amount of heat produced by eddy currents circulating in the laminated iron of the core.

The cycle experiments were made in the following manner, which brought to light the creeping above referred to. The ring of iron to be experimented on was wound with two magnetizing coils superimposed. A constant magnetizing current sufficient to produce the extreme negative induction was maintained in one of these, so that by simply making and breaking the other circuit, thereby starting and stopping an opposite current of any chosen amount, a rapid magnetic cycle could be obtained passing from and returning to the induction proper to the constant current.

Supposing it was desired to obtain a point on the part of the

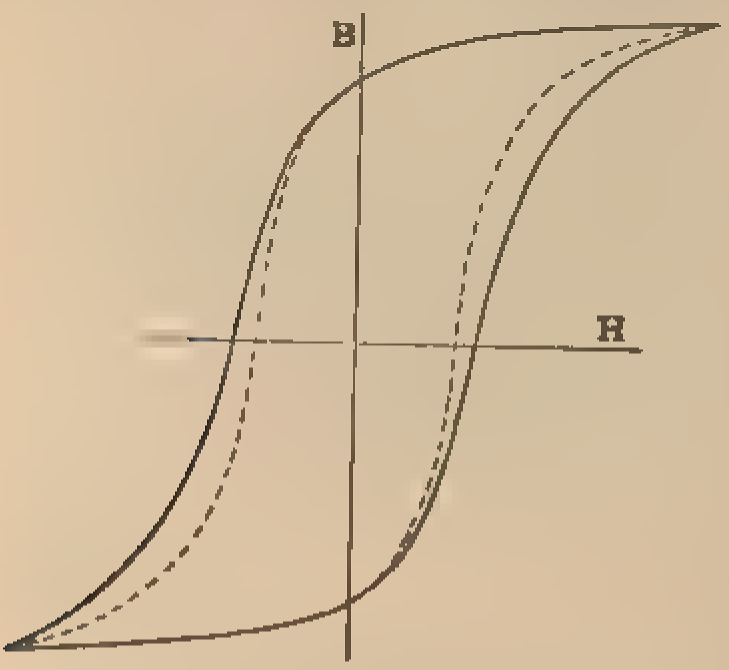

Fig. 170.

curve rising from the extreme negative induction, it was only necessary to apply the proper positive current and read off the change of induction on the ballistic galvanometer. This subtracted from the original value of the induction gave the induction remaining. The time necessary to do this was about ½ of the period of the needle, or nearly a second. On the other hand to obtain a point in the falling curve it was necessary to apply the maximum positive current, and then diminish to the value required. For this a longer time, about half a minute, was necessary.

[redacted]

Hysteresis Loss practically Constant for all except very Slow Cycles.

Working thus the authors found a difference between the steepness of the rising and falling curves, which revealed a distinct creeping down of the magnetization in the falling curve, occurring in the longer interval required for the operations. Thus a very slow and a very rapid cycle between the same extreme inductions ought to have the shapes shown in Fig. 170 by the dotted and full curves respectively. The difference is about that between the cycles for periods of two seconds and half a minute respectively.

The general result of Messrs. Evershed and Vignoles' experiments is to show that there is very little difference between the energy lost in hysteresis at periods of from 2 seconds to 1/100 of a second; according to the authors' estimate the utmost difference is not more than 4 or 5 per cent., and probably less than 2 per cent. For very slow cycles however in periods of several minutes the energy lost is from 20 to 25 per cent. less.

Messrs. Ayrton and Sumpner's experiments, which were performed by the method of testing transformers described above, show that the iron losses are constant for all loads, and do not change to more than a slight extent with alteration of frequency. The main results obtained with 4½ kilowatt closed circuit transformers (Mordey type) were :—

(1) The greater the frequency the greater the efficiency for any particular load.

(2) The greater the load the greater the efficiency for the same frequency.

Specimens of the actual numbers are given in the following table.

Load (Watts).	Frequency. 100	120	160
	Efficiency.		
1000	84·46	85·59	88 58
2000	92·19	92·76	93·84
3000	94·27	94·79	98·30
4000	95·22	95·63	98·91

Ewing's Magnetic Curve-Tracer. Description.

An instrument for tracing hysteresis curves has been invented by Prof. Ewing, and is illustrated diagrammatically in Fig 171.* A mirror E is pivotted so as to turn on a needle point, about a vertical or a horizontal axis A wire BB is stretched horizontally along the narrow gap between the pole-faces of a constant electromagnet made of a piece of iron pipe slit along a generating line. This wire carries the magnetizing current and therefore experiences an electromagnetic force proportional to that current tending to move it across the lines of magnetic force between the pole-faces where it is situated, against a return force due to the stretching weight nearly proportional to the displacement. Thus a displacement of the mirror round a vertical axis proportional to the magnetizing force is produced.

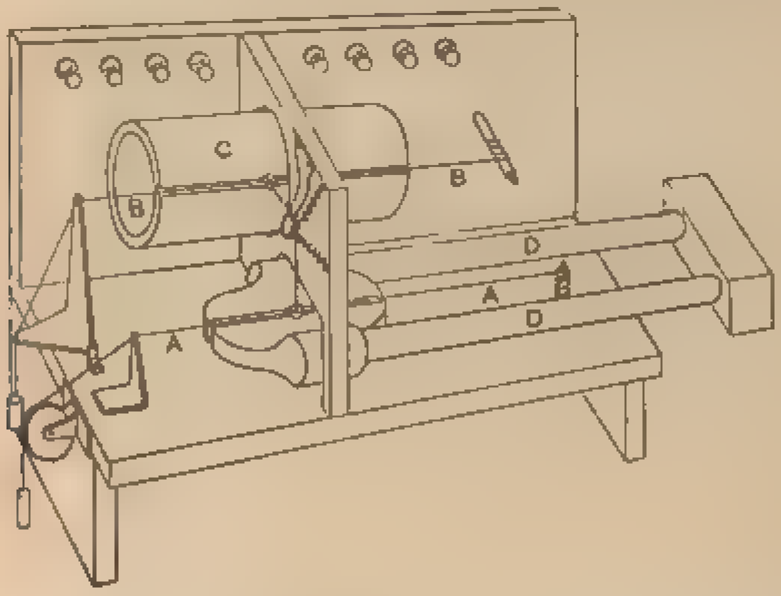

Fig. 171.

Another wire AA is stretched along the narrow space between the two long pole-faces of the electromagnet DD. The rods DD are made of the substance to be examined, and at one end are united by a yoke of soft iron, at the other terminated by suitable soft iron pole-pieces, and are surrounded by coils in which flows the current which also passes through BB. A constant current flows in AA, which is therefore acted on by vertical electromagnetic forces proportional to the induction

* See *Electrician*, Aug. 12, 1892, for specimens of curves drawn by the machine.

in *DD*, and resisted by the tension of the wire. This gives a vertical displacement proportional to the induction in the iron specimens.

A ray of light reflected from the mirror therefore traces on a screen a curve of magnetization, and if a cycle of magnetization is repeatedly produced by an alternating current the corresponding closed loop is described on the screen by the spot of light, and by the persistence of impressions on the retina remains visible so long as the cycle is described. It can therefore be drawn or photographed at pleasure for comparison.

The moving parts of the apparatus it will be noticed have extremely little mass, and it is stated that even with a period of $\frac{1}{10}$ or $\frac{1}{20}$ of a second the action of the instrument is but little affected by inertia.

Results of Experiments with Curve-Tracer. Experimenting with bars of iron of different thicknesses Prof. Ewing has found by means of his curve-tracer that, while with thin bars and laminated iron generally the cycle has the form shown in Figs. 164, 165, 166, with comparatively sharp corners at the turning points which are also the places of extreme induction, with thicker bars the turning points are rounded off and the maximum inductions are well inside the ends of the curve. With cycles of frequency two or three or more per second, the cycle becomes a figure resembling closely an ellipse with its major axis inclined to the axis of abscissæ.

This is an effect which, as noticed by Hopkinson, is produced by eddy currents. According to Ewing it is probably produced also by creeping of the magnetization from the surface inwards.

Searle's Magnetic Curve-Tracer Another form of magnetic curve-tracer invented by Mr. G. F. C. Searle * is represented in Fig. 172. A thin wire of aluminium *AB* about 80 cms. long has attached near its upper end a horizontal needle *C*. At the lower end it carries a fork of aluminium, the prongs of which are connected by a silk fibre *DE*. To this fibre a light mirror *F* is attached by wax, and carries a small needle the length of which is at right angles to the fibre. A disk of mica about an inch in diameter is attached to the lower edge of the mirror. The undisturbed position of the mirror is vertical, and therefore that of the mica disk horizontal. A piece of cardboard is placed in a horizontal position close below the disk, and hence when the latter is moving gives rise to a damping action which soon reduces the mirror to rest. A mica vane attached to the vertical wire *AB* damps the motion round the vertical axis.

* *Proc. Camb. Phil. Soc.* May 16, 1892.

Action of Instrument.

It will be seen that the mirror has two distinct freedoms of motion (1) round a vertical axis, (2) round the horizontal fibre as axis. The apparatus is set up with the magnet AC in the magnetic meridian and a coil carrying the magnetizing current is placed in the magnetic meridian with its axis passing through the centre of C. The specimen of iron to be tested is placed vertically in a magnetizing coil, with its upper end nearly opposite the mirror F, and in the east and west (magnetic) plane through its centre. A compensating coil is used to annul the direct effect of the magnetizing solenoid, and the magnetization of the iron specimen then tends to tilt the magnet on the mirror, while the magnetizing current turns it round a vertical axis by

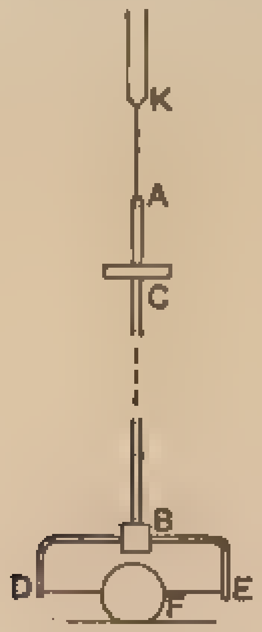

Fig. 172.

acting on the upper needle C. The controlling force on the upper needle is the horizontal intensity of the field there, that on the lower is partly gravity, due to the mirror and attached disk being suspended with their centre of gravity a little below the silk fibre, and partly the earth's vertical force. The lower needle is of course placed with its north pointing end down.

The instrument is hung within a case, and a lamp and scale, with lens forming a window in the case through which both incident and reflected rays pass, is employed to give a bright spot on a screen. [It was found that no trouble was experienced from the silk fibre, as the spot of light after deflection returned

within $\frac{1}{50}$ of an inch to its former position and the zero seemed to be permanent.]

The effect of any specimen (if it is taken sufficiently long) is proportional to its intensity of magnetization **I**, and for iron this is not very different from $\mathbf{B}/4\pi$, so that the ordinates of the curve described by the spot of light may be taken as proportional to $\mathbf{B}a/4\pi$, where a is the area of cross-section, and the abscissæ as proportional to the magnetizing force **H**.

Standardizing of Instrument.

The indications of the instrument can easily be reduced to absolute measure, by noting (1) the deflection of the spot of light produced by a known current, and calculating the corresponding field intensity in the magnetizing solenoid, (2) by ascertaining the deflection of the mirror round the horizontal axis produced by placing a magnetized steel wire in position in the solenoid (of course without current) noting the deflection and then determining the magnetic moment of the wire.

Fig. 173.

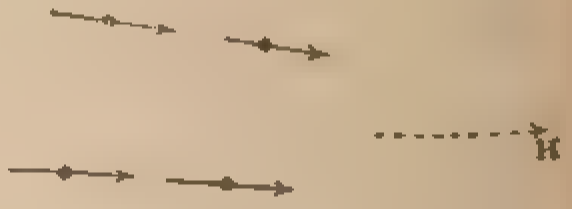

Fig. 174.

Ewing's Molecular Theory of Magnetization.

Much light is thrown on the nature of magnetization by a molecular theory recently put forward by Ewing.* In this theory the actions of the members of a group of small magnets on one another are studied, and show that in all probability the peculiar character of the curves of magnetization found for iron, and many of the complex phenomena of effects of stress and temperature are explicable by the action of the molecular magnets on one another. We may suppose that the molecular magnets in a piece of iron are stably arranged in a regular order, but so that the external magnetic force exerted by them is zero. Such a group would be the four small magnets with centres at the corners of a square, represented by the arrows in Fig. 173. If then a magnetic force **H** is applied to them they will take up the positions shown in Fig. 174, if the force be

* *Phil. Mag.* Sept. 1889

great enough. But if the force be small there will only be a slight displacement of the magnets towards concurrence with **H**, but if the force is gradually increased this concurrence will become gradually more marked, until **H** becomes too great to be resisted by the mutual actions of the particles, and their equilibrium becoming unstable, they suddenly swing round

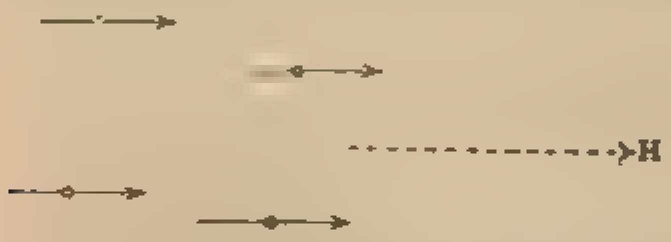

FIG. 175.

towards parallelism with **H**, each remaining however still at finite angle with the direction of the magnetic force. As this is still further increased, this angle becomes gradually smaller and the magnets approach to parallelism with one another and with **H** as shown in Fig. 175.

Thus the group becomes magnetized in the direction of **H**, at

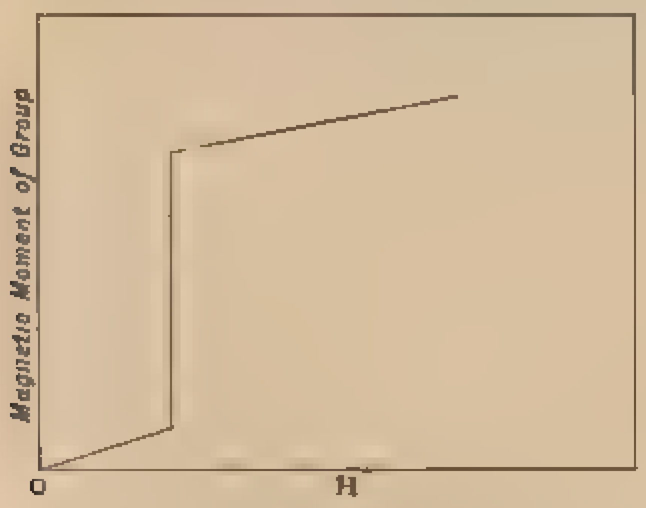

FIG. 176

Curve of Magnetization of Group of Small Magnets.

first slowly, then suddenly, and again slowly, as represented in Fig. 176. If the system consisted not of a single simple group such as this, but of a large number of different groups, the sharp angles would be rounded off and we should get the actual curve of magnetization. Again, on gradual withdrawal and reversal

of the magnetic force the curve of diminution of magnetic moment would not coincide with the curve for increase, and zero value of **H** would be reached with a certain residual magnetic moment, in the direction of the applied force. On reversal and gradual increase of the force, the small magnets would again become unstable and would suddenly swing round now in the opposite direction.

Hysteresis Cycle.

In fact a hysteresis cycle would be obtained. Such curves and cycles have been drawn for complex groups of small magnets arranged at first so as to have nearly zero external effect within a large solenoid, by gradually increasing the current in the solenoid from zero to a considerable value, diminishing it through zero to a negative value, and so on as in an ordinary cycle of magnetization, and measuring the magnetic moment of the system by means of a magnetometer in the ordinary way. These curves agree wonderfully with those given by actual specimens of iron. The physical cause of the dissipation of energy in hysteresis Ewing conjectures to be the development of eddy currents in the surrounding medium in consequence of the oscillation of the small magnets about their new positions when displaced by the magnetic force. For a full account of this theory and its consequences the reader is referred to Ewing's paper *loc. cit.* and to his book on *Magnetic Induction in Iron and other Metals*.

Section IV

DETERMINATION OF VERDET'S CONSTANT

Verdet's Constant.

Verdet's constant has been defined at p. 226 above as the amount of turning of the plane of polarization of a ray of plane polarized light per unit difference of magnetic potential between the extremities of the portion of the ray considered. Determinations of this important constant have been made by J. E. H. Gordon,* Henri Becquerel,† and Lord Rayleigh.‡

Lord Rayleigh's Determination.

In Lord Rayleigh's experiments, of which we give here a short account, a beam of light from sodium burning in the flame of a Bunsen lamp, *A* (Fig. 177), intensified by a jet of

Fig. 177.

oxygen made to play round the flame, passed through a slit in a screen, *B*, in front of the flame to a direct vision prism, *D*, thence through a collimating lens, *E*, to a Nicol's prism, *F*. The plane polarized beam emerging from *F* was received by a syrup-cell polarimeter, *G*, and then passed through a tube, *H*, filled with bisulphide of carbon, and a slit in the screen, *I*, to the analyzing prism, *J*.

Details of Apparatus.

The screen, *B*, was made of looking-glass, and the slit created by removing a narrow strip of the silvering. By this arrangement, together with a parallel mirror, *C*, a considerable increase of illumination was obtained.

The direct vision prism was used to purify the light from rays of other refrangibilities than that of sodium. The lens *E*

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* *Phil. Trans. R. S.* 1877, p. 1.
† *Ann. de Chimie*, 1882.
‡ *Phil. Trans. R.S.* Pt. II. 1885.

rendered the rays parallel before incidence on the polarizer. The polarimeter *G* was a cell containing a stratum of strong sugar syrup, made according to Poynting's plan, so that in one half the thickness of the stratum was the full width of the cell, in the other half was diminished by a plate of glass, so that a difference of rotation of about 2° was produced by the two halves.

Two tubes of brass, one 31·591 inches long and 1½ inch in diameter, the other 29·765 inches long and 1 inch in diameter, and closed at the ends with plates of glass, were used to contain the bisulphide of carbon. The temperature of the liquid was observed on a thermometer inserted in an opening near one end of the tube.

The analyzer was in some of the experiments a Nicol, in others a double image prism, and was mounted in the usual way on a graduated circle.

Precautions for Accuracy in Use of Analyzer.

It is necessary for accuracy in reading the amount of rotation that the axis round which the analyzer turns should coincide in direction with the ray. This adjustment was made by observing the direction of the ray by means of a telescope with cross wires, and then replacing the telescope by the Nicol or double image prism. Error however was introduced by the passage of heat into the liquid, whereby the upper part of it became slightly warmer than the lower. To eliminate this to a first approximation the two positions of the Nicol, nearly 180° apart, which gave equality of illumination in the two parts of the field, were read off. It was found that by the use of a double image prism, read in four positions nearly 90° apart, the error could be more nearly got rid of.

The adjustment of the analyzer to an exact match between the two halves of the field was facilitated by arranging an auxiliary coil round the tube, in circuit with a Leclanche cell, worked by a reversing key within reach of the observer at the analyzer. With this the plane of polarization could be rocked "backwards and forwards through a small angle about its normal position. The amount of the rocking being suitably chosen, the comparison of the three appearances (two with auxiliary current and one without) serves to exclude some imperfect matches that might otherwise have been allowed to pass."*

Arrangement of Magnetizing Helix.

The helix was wound on an ebonite tube placed round the bisulphide carbon tube, from which it was separated by several

* Lord Rayleigh, *Proc. R. S.* Vol. xxxvii. (June 19, 1884).

layers of paper. The length of this tube between the end flanges was 9·990 inches. The winding was performed with two wires side by side, so that by sending a current in opposite directions through them, or by trying to force a current from one to the other, the insulation of the coil could be tested, while both wires could be used to produce the magnetic field. The coil was found on test to insulate satisfactorily. The internal diameter was 2·188 inches, the external 4·13 inches. The number of turns was 3,684.

Calculation of Difference of Potential.

By (30), p 263 above, the potential at any point L on the axis external to the coil is $V_1 - V_2$, where V_1, V_2 are the potentials at the point due to the ends. But if we consider another point M on the axis on the side of the coil remote from L, it follows from p. 263 that the difference of magnetic potential, $\Omega - \Omega'$, between L and M taken along the axis through the coil is for unit current given by

$$\Omega - \Omega' = 4\pi N - \{V_1 - V_2 - (V'_1 - V'_2)\} \quad . \quad . \quad (43)$$

where the dashed letters refer to M, and the undashed to L.

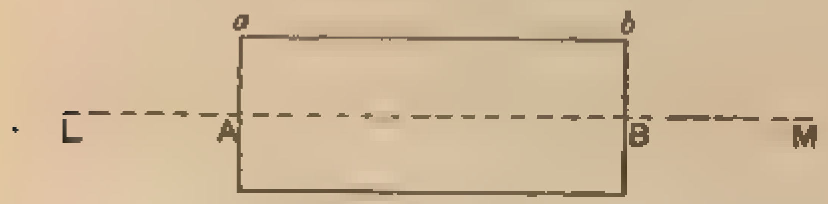


FIG. 178.

But if n be the number of turns per unit of length in any layer of radius Aa or Bb (Fig. 178), r_1, r_2, x_1, x_2, the distances La, Lb, LA, LB, respectively, the potential at L due to the ends of the layer is, if the end B be taken as positive, that at A negative,

$$2\pi n\{r_2 - x_2 - (r_1 - x_1)\}.$$

If a be the radius of the layer (supposed small in comparison with x_1 or x_2) we have approximately

$$r_1 = x_1\left(1 + \frac{1}{2}\frac{a^2}{x_1^2} - \frac{1}{8}\frac{a^4}{x_1^4}\right)$$

$$r_2 = x_2\left(1 + \frac{1}{2}\frac{a^2}{x_2^2} - \frac{1}{8}\frac{a^4}{x_2^4}\right).$$

Hence the potential of the layer at L due to the ends is

$$\pi n(x_2 - x_1)\left\{\frac{a^2}{x_1x_2} - \frac{a^4}{4}\,\frac{x_2^2 + x_1x_2 + x_1^2}{x_1^3x_2^3}\right\}.$$

Calculation of Potential.

To sum the effects of all the layers the mean values of a^2 and a^4 in the last expression are to be substituted for these quantities, and the expression multiplied by the number of layers. But the mean value of a^2 is, if a_1, a_2, be the internal and external radii,

$$\frac{1}{a_2 - a_1}\int_{a_1}^{a_2} a^2 da = \frac{a_2^3 - a_1^3}{3(a_2 - a_1)}$$

and in the same way the mean value of a^4 comes out

$$(a_2^5 - a_1^5)/5(a_2 - a_1).$$

Thus if there are n' layers, the total number of turns in the coil is $nn'(x_2 - x_1)$, and therefore the potential at L due to the ends of the coil is

$$V_1 - V_2 = \pi N\left\{\frac{a_2^3 - a_1^3}{3(a_2 - a_1)}\,\frac{1}{x_1x_2} - \frac{a_2^5 - a_1^5}{20(a_2 - a_1)}\,\frac{x_1^2 + x_1x_2 + x_2^2}{x_1^3x_2^3}\right\}.$$

In the same way we should find for M, putting $MA = x'_1$, $MB = x'_2$ (numerically, so that $x'_1 - x'_2 = x_2 - x_1$), that the potential at M due to the ends of the coil is

$$V'_1 - V'_2 = -\pi N\left\{\frac{a_2^3 - a_1^3}{3(a_2 - a_1)}\,\frac{1}{x'_1x'_2} - \frac{a_2^5 - a_1^5}{20(a_2 - a_1)}\,\frac{x'^2_1 + x'_1x'_2 + x'^2_2}{x'^3_1x'^3_2}\right\}.$$

Value of Difference of Magnetic

With these values of $V_1 - V_2$, $V'_1 - V'_2$ (43) becomes

$$\Omega - \Omega' = 4\pi N\left[1 - \left\{\frac{a_2^3 - a_1^3}{12(a_2 - a_1)}\left(\frac{1}{x_1x_2} + \frac{1}{x'_1x'_2}\right)\right.\right.$$

$$\left.\left. - \frac{a_2^5 - a_1^5}{80(a_2 - a_1)}\left(\frac{x_1^2 + x_1x_2 + x_2^2}{x_1^3x_2^3} + \frac{x'^2_1 + x'_1x'_2 + x'^2_2}{x'^3_1x'^3_2}\right)\right\}\right] \quad (43')$$

Potential for Ends of Tube.

In the apparatus used

$$a_2 = 2{\cdot}065, \quad a_1 = 1{\cdot}094, \quad a_2 - a_1 = {\cdot}971,$$

in inches so that

$$\frac{a_2^3 - a_1^3}{12(a_2 - a_1)} = \cdot 6433, \quad \frac{a_2^5 - a_1^5}{80(a_2 - a_1)} = \cdot 4622.$$

Data of Apparatus.

Further for the longer tube in inches

$$x_1 = x'_2 = 10\cdot 800,$$

$$x_2 = x'_1 = 20\cdot 790.$$

and for the shorter

$$x_1 = x'_2 = \ \ 9\cdot 887,$$

$$x_2 = x'_1 = 19\cdot 877.$$

Hence the required difference of potential between the ends of the tube was for unit current

$$= 4\pi N(1 - \cdot 00573 + \cdot 00006) = 4\pi N \times \cdot 99433$$

for the first tube, and

$$4\pi N(1 - \cdot 00655 + \cdot 00008) = 4\pi N \times \cdot 99353$$

for the second.

Mode of Measuring Current.

To measure the current the difference of electric potential between the terminals of a coil of resistance R, placed in the circuit of the helix, was balanced by the electromotive force of a derived circuit in which was placed a Clark cell. At 15° the resistance of the coil was 1·4958 B.A. unit. Currents of about 1 ampere were used in the experiments.

We must refer the reader to Lord Rayleigh's paper, *loc. cit.*, for further particulars of the results obtained and their treatment.

Final Result.

It was found as the mean result of a large number of experiments, that if x denote the value of Verdet's constant in minutes of angles for carbon disulphide at 18° C.,

$$x = \cdot 04202.$$

Variation of Rotation with Temperature

The rotation in bisulphide of carbon according to Bichat varies with the temperature (Centigrade) according to the formula $1 - \cdot 00104t - \cdot 000014t^2$, and this was used to correct for deviation of temperature of the bisulphide from 18°.

Becquerel's result was

$$x = \cdot 0452.$$

Comparison with Gordon's and Becquerel's Results.

Gordon's experiments were made for light corresponding to the thallium line. The value obtained was ·05238 minute for this light at a temperature about 13° C. Taking the rotation as proportional to $\mu^2(\mu^2 - 1)\lambda^{-2}$, where μ is the refractive index and λ the wave length, Gordon's result would give for sodium light

$$x = \cdot 04163$$

about 13° C., or

$$x = \cdot 0413$$

at 18°.

CHAPTER XIV

ELECTRIC OSCILLATIONS AND ELECTRIC RADIATION

SECTION 1

EXPERIMENTS ON ELECTRIC OSCILLATIONS

Feddersen's Experiments on Leyden Jar Discharges.

THE approximate truth of the theory of electric oscillations given above was shown experimentally by Feddersen and others who observed the spark of a Leyden jar discharge in a rotating mirror In Feddersen's experiments the mirror was driven at a known rate by clockwork, which also at a certain definite position of the mirror started the discharge of a battery of Leyden jars across the spark-gap of a micrometer. The mirror was concave, and the spark-gap was so situated that an image of the spark was thrown by the mirror on a ground-glass plate, for which a photographic plate could be substituted

Effect of Length and Resistance of Discharging Conductor.

The spark when taken with a short metallic connection to the micrometer and between metallic points was a long tapering band of light beginning at the broad end with a clear white light and then fading off and narrowing through a greenish colour to red at the narrow end. No doubt the bright part was the spark proper, and the red part was due to the cooling gases and particles of metal.

As the resistance of the discharging circuit was increased without increase of length, the red part disappeared and a bright line with projecting luminous bands at its upper and lower ends took its place, and this in its turn gave place to a succession of bright lines at gradually increasing distances apart, indicating an intermittent discharge.

Oscillatory Discharge.

When the discharging arc was long and at the same time of sufficiently low resistance, the image of the spark as seen on the glass plate or as photographed became a succession of equidistant transverse bands each shading off at the sides into dark spaces, separating it from the next on either side.

* *Pogg. Ann.* 112, 113 (1861).

Effect of Increase of Capacity. It was found by experimenting with different numbers of jars in the battery and varying the resistance until the discharge just ceased to be oscillatory, that the limiting value of the resistance was inversely proportional to the square root of the charged surface of the battery, that is to the square root of the capacity, and independent of the spark-length, that is of the difference of potential. This agrees with the theory given above (p. 188), according to which the limiting value of the resistance R is $2\sqrt{L/C}$.

It was also found that if the discharge was from a battery of surface S to an uncharged battery of surface S', the period was proportional to $\sqrt{SS'/(S+S')}$, which also obviously agreed with theory.

Again, when the length of the discharging arc was increased the period was increased also, though not entirely owing to the increase in resistance, and the arrangement of the discharging wire was found very materially to affect the period in a way clearly depending on the value of the induction.

Experiments were also made by von Oettingen, Knochenhauer, Riess, Helmholtz, and others confirmatory of the theory. A full account of these researches will be found in Wiedemann's *Elektricität*, Band IV. pp. 177 *et seq.*

Schiller's Experiments. We shall only notice further of these earlier researches on electric oscillations the electric experiments of Schiller.* In these the oscillating discharge took place through a coil joining the plates of a condenser, and no spark was produced. The

Mode of Experimenting. current induced in the secondary of an induction coil by the breaking of the primary circuit by an arrangement of contact levers and pendulum interruptor, was used to charge the condenser, which gave rise to electric oscillations in the coil connecting the plates. One plate of the condenser and one terminal of the secondary were connected to the insulated pair of quadrants of an electrometer.

At an interval (which could be adjusted at pleasure by properly arranging the contact levers and interruptor) after the primary circuit was broken, the secondary was also broken and the electrometer detached from the condenser plate. By varying the interval and making repeated observations the potential at different stages of the oscillation could be read off and the period ascertained.

The time interval was found by sending a current through one coil of a differential galvanometer accurately adjusted for

* *Pogg. Ann.* 152 (1874).

balance, and an equal current through the other coil at the next break, and observing the deflection. From this the time of flow of the current could be easily calculated.

Measurement of Time Interval. Elimination of Electrostatic Capacity of Coil.

The coil of course had an electrostatic capacity of its own which it was impossible to calculate. This was determined by joining it with various combinations of six different condensers, and the periods of oscillation were observed and the relative capacities of the separate condensers calculated. The results were compared with those obtained by joining each condenser singly with the coil and observing the period of oscillation. The results were found to agree exceedingly well.

This method was applied to determine the damping of the oscillations with different dielectrics connecting the plates, and thence to find the resistances of the substances. It was also used to determine the relative capacities of the same arrangement of plates with different dielectrics, and hence to find the specific inductive capacity.

Determination of Specific Inductive Capacity by Oscillations.

Thus if T denote the period of oscillation with the coil alone, T_1 or T_2 the period with the condenser, according as air or the dielectric in question was between the plates, C_1, C_2 the capacities of the condenser in these cases respectively, then by the value of the period given at p. 188, if the resistance be neglected, we have

$$K = \frac{C_2}{C_1} = \frac{T_2^2 - T^2}{T_1^2 - T^2}.$$

By this formula the values given in the table were found from the experiments:—

Results.

	K		K
Ebonite	2·21	Paraffin—	
Caoutchouc—		Quickly cooled . .	1·68
Pure	2·12	Slowly cooled . .	1·81 1·89
Vulc.	2·69	Flint Glass	5·78 to 5·88

The half period in these experiments varied from ·000056 sec. to ·00012 sec. Considerably larger values of K were found for the same specimens by Siemens' method of successive charge and discharge (see Vol. I. p. 448).

The results calculated by theory agreed exceedingly well with those found by experiment. There can be no doubt that the theory is very imperfect for many reasons. For example the current in the coil owing to the varying electrostatic capacity of its different parts could not be the same throughout at any one instant, and much more so without a condenser than when one of considerable capacity was attached to its terminals.

Section II

ELECTROMAGNETIC RADIATION

We now give in conclusion some account of the remarkable verification of Maxwell's theory of Electrical Radiation lately given by Hertz. We give first his solution of the problem of the propagation of electromagnetic waves from a source symmetrical about an axis, in order that the experimental results may be more easily understood.

Vibrating Electric Doublet. As source of the waves we take an electric doublet, that is two equal and opposite electric charges concentrated at two points infinitely near to one another, or, more properly, at a distance apart infinitely small in comparison with the distance from either charge of any point at which the electric or magnetic force is considered. The moment of the doublet, that is the product of either charge into the distance between the two points, we shall suppose to vary, by variation of the charges only, as a simple harmonic function of the time; but its maximum value will be supposed finite and constant. Such a source is the exact electric analogue of the infinitely short magnet considered at p. 8 above.

Hertz's "Dumb-bell" Vibrator. Such a source may be regarded as physically realized, except for points very near it, by two equal and oppositely charged spheres connected by a straight conductor. This was the form of electric vibrator employed by Hertz in some of his most

important experiments. The spheres were charged to opposite potentials by an induction coil, and then discharged into one another, setting up thereby, as the discharge was oscillatory, electromagnetic waves in the surrounding medium, which were propagated outwards from the vibrator in all directions. The existence of these waves was detected by a simple receiver, or *resonator* as it has been called, consisting of a circle of wire, complete with the exception of a very small spark-gap between two small knobs which tipped the ends of the wire, and properly placed with reference to the vibrator.

The calculation of the electric and magnetic forces at points at a distance from such a vibrator comparable with its dimensions would be very difficult, but for points at distances very great in comparison with the distance between the centres of the spheres, the forces must be very approximately the same as those due to the electric doublet.

Theory of Vibrating Doublet.

In what follows we shall take the origin at the point midway between the two charges of the doublet, and the axis of the doublet as axis of z. Since everything is symmetrical round this axis, we need consider only the disturbance at any instant at a point distant z from the origin and ρ from the axis. We shall call the plane through the origin at right angles to the axis the equatorial plane, and any plane through the axis a meridian plane. As starting-point of the solution we use equations (27), (27′), (28), (28′) of p. 201 above.

The electric forces obviously lie in meridian planes, and the lines of magnetic force must be circles round the axis of the system. Thus $\gamma=0$, and (28) becomes

$$\frac{\partial \alpha}{\partial x}+\frac{\partial \beta}{\partial y}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Equations of Motion for Doublet.

We might transform these equations to cylindrical co-ordinates and then proceed to find the solution of the problem for the case supposed. The following process is simpler.

Equation (1) shows that $\alpha dy-\beta dx$ is a complete differential of some function of y, x. Using Hertz's notation we take this function as $\partial \Pi/\partial t$. Thus

$$\alpha=\frac{\partial^2 \Pi}{\partial t \partial y},\ \beta=-\frac{\partial^2 \Pi}{\partial t \partial x} \quad . \quad . \quad . \quad . \quad (2)$$

and equations (27) of p. 201 become

$$\left.\begin{aligned} K\frac{\partial P}{\partial t} &= \frac{\partial^3\Pi}{\partial t\partial x\partial z} \\ K\frac{\partial Q}{\partial t} &= \frac{\partial^3\Pi}{\partial t\partial y\partial z} \\ K\frac{\partial R}{\partial t} &= -\frac{\partial}{\partial t}\left(\frac{\partial^2\Pi}{\partial x^2}+\frac{\partial^2\Pi}{\partial y^2}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

Now equations (3) declare that the quantities

$$KP - \partial^2\Pi/\partial x\partial z,\ KQ - \partial^2\Pi/\partial y\partial z,\ KR + \partial^2\Pi/\partial x^2 + \partial^2\Pi/\partial y^2$$

are independent of t. Their values cannot therefore have any influence on the wave propagation, and may be taken as zero in each case. Thus we assume

$$\begin{aligned} KP &= \frac{\partial^2\Pi}{\partial x\partial z} \\ KQ &= \frac{\partial^2\Pi}{\partial y\partial z} \\ KR &= -\left(\frac{\partial^2\Pi}{\partial x^2}+\frac{\partial^2\Pi}{\partial y^2}\right) \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

These equations fulfil, it will be seen, equation (3). Substituting now in (27′), p. 201, we find

$$-\mu\frac{\partial a}{\partial t} = -\mu\frac{\partial^3\Pi}{\partial t^2\partial y} = -\frac{1}{K}\frac{\partial}{\partial y}\left(\frac{\partial^2\Pi}{\partial x^2}+\frac{\partial^2\Pi}{\partial y^2}+\frac{\partial^2\Pi}{\partial z^2}\right),$$

that is

$$\frac{\partial}{\partial y}\left(\frac{\partial^2\Pi}{\partial t^2}-\frac{1}{K\mu}\nabla^2\Pi\right) = 0,$$

and similarly

$$\frac{\partial}{\partial x}\left(\frac{\partial^2\Pi}{\partial t^2}-\frac{1}{K\mu}\nabla^2\Pi\right) = 0$$

These show that the quantity in the brackets is a function of z and t. Thus

$$\frac{\partial^2\Pi}{\partial t^2}-\frac{1}{K\mu}\nabla^2\Pi = f(z,t) \quad . \quad . \quad . \quad . \quad (5)$$

Differential Equation of Oscillating Electric Doublet

That we may put $f(z, t) = 0$ without affecting the electric and magnetic fields may be seen in the following manner. Let $\Pi = \phi(x, y, z, t)$ be a solution of (5). Then since the electric and magnetic forces involve differentiation with respect to x and y we may use instead of this the value $\phi(x, y, z, t) + \psi(z, t)$ for Π, where $\psi(z, t)$ is *any* function of z and t. This would give the differential equation

$$\frac{\partial^2\Pi}{\partial t^2} - \frac{1}{K\mu}\nabla^2\Pi = f(z, t) + \chi(z, t)$$

where $\chi(z, t)$ arises from $\psi(z, t)$. Since $\psi(z, t)$ may be anything we please we may choose it so that $\chi(z, t) = -f(z, t)$. Hence putting $f(z, t) = 0$, in (5), will not affect the electric or the magnetic force at any point. Thus we have for the equation of propagation of electromagnetic disturbance

$$\frac{\partial^2\Pi}{dt^2} = \frac{1}{K\mu}\nabla^2\Pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5')$$

The well-known general solution of this equation is

$$\Pi = \frac{1}{r}\{F_1(r - vt) + F_2(r + vt)\} \quad . \quad . \quad . \quad . \quad (6)$$

where $v = 1/\sqrt{K\mu}$; and F_1, F_2 are arbitrary functions.

To find a solution adapted to the vibrator we have imagined, we put

$$\Pi = \frac{\Phi}{r}\sin(mr - nt) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where $m = 2\pi/\lambda$ (λ = wave-length) and $n/m = v = 1/\sqrt{K\mu}$. This satisfies the differential equation and is of the form (6). The values of P, Q, R derived from it satisfy (28), p. 201, and we shall see that it is applicable to the present case.

Solution of Equation.

First as the case supposed is that of an electric doublet, the moment Φ of which varies as a simple harmonic function of the time, the field of electric force in the immediate neighbourhood of the doublet, that is, at any point whose distance from the doublet is a small fraction of the wave-length of the disturbance, must at each instant precisely correspond to the field of a small magnetic doublet of the same moment numerically as that which the electric doublet has at that instant. Now the lines of force for this case are discussed at p. 8, and illustrated in Fig. 2.

The magnetic force is there derived from a potential of the form

$$V = -m\frac{\partial}{\partial x}\frac{1}{r},$$

where m is the moment of the magnetic doublet.

Comparison of Solution with Magnetic Doublet.

If instead of m we write $\Phi \sin nt$, we see that the electric force to correspond ought to be given by a potential

$$V = -\Phi \sin nt \frac{\partial}{\partial z}\left(\frac{1}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

But (7) may be written

$$\Pi = \frac{\Phi \sin nt}{r} + \frac{\Phi}{r}\{\sin(mr - nt) + \sin nt\},$$

and the second term on the right is practically zero when mr $(= 2\pi r/\lambda$, where λ is the wave-length) is a very small angle, that is when r is small in comparison with the wave-length. Thus in the immediate neighbourhood of the vibrating doublet

$$\Pi = \frac{\Phi \sin nt}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and, by (4), the potential V is given by

$$V = -\frac{\partial \Pi}{\partial z} = -\Phi \sin nt \frac{\partial}{\partial z}\left(\frac{1}{r}\right) \quad . \quad . \quad . \quad . \quad (10)$$

which agrees exactly with the magnetic analogue. This may be taken as so far a verification of the solution.

Further (7) gives electric force, and therefore also magnetic force, everywhere zero at an infinite distance, which must be the case also for physical reasons.

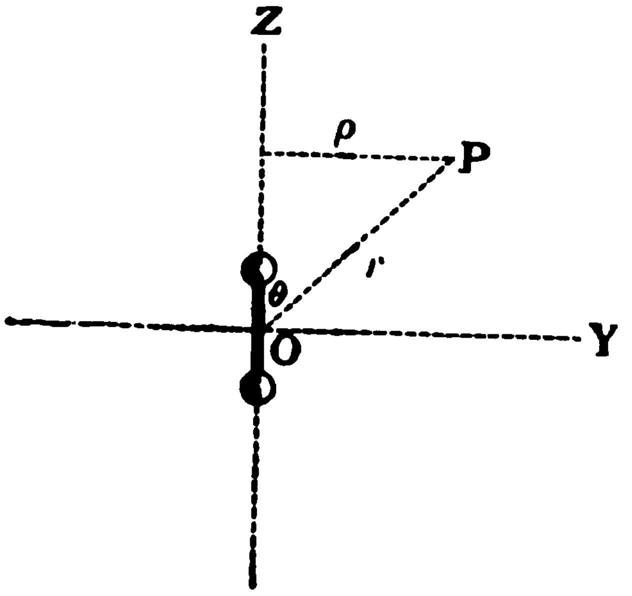
Solution Modified for Symmetry round Axis.

Since the field must be symmetrical about the axis of z, the electric force lies everywhere in a meridian plane. It will be sufficient therefore to use co-ordinates z along the axis, and ρ at right angles to it, and to calculate for any point (z, ρ) the components of electric force perpendicular and parallel to the axis, and that of magnetic force parallel to the meridian plane. We shall take the meridian plane considered as plane

of y, z, so that ρ (Fig. 179)* becomes identified with y, and denote the components specified above by Q, R, a respectively, putting P, β, $\gamma = 0$. In calculating these components by (7) we have to put $r^2 = z^2 + \rho^2$, and in finding R to write the third equation of (7) in the form

Expressions for Electric and Magnetic Forces.

$$KR = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Pi}{\partial\rho}\right)$$



The figure represents an electric oscillator at the origin of co-ordinates as intended to be understood by Hertz.

FIG. 179.

which is its proper form for the case of symmetry round the axis of z. Thus we find

$$\left.\begin{aligned}
KQ &= \frac{2\Phi}{r^3}\left\{\left(1 - \frac{m^2r^2}{2}\right)\sin(mr - nt) - mr\cos(mr - nt)\right\}\sin\theta\cos\theta \\
KR &= \frac{\Phi}{r^3}\,[2\{\sin(mr - nt) - mr\cos(mr - nt)\} \\
&\qquad - \{3\sin(mr - nt) - 3mr\cos(mr - nt) \\
&\qquad\qquad - m^2r^2\sin(mr - nt)\}\sin^2\theta] \\
a &= \frac{n\Phi}{r^2}\{mr\sin(mr - nt) + \cos(mr - nt)\}\sin\theta
\end{aligned}\right\} \quad (11)$$

* This figure is taken from Dr. Lodge's translation of Hertz's paper in *Nature*, Vol. XXXIX. (1888—9).

Forces at points near the Vibrator.

At points very near the vibrator we get from these equations

$$\left.\begin{aligned} KQ &= \frac{2\Phi}{r^3}\sin(mr - nt)\sin\theta\cos\theta \\ KR &= -\frac{\Phi}{r^3}\sin(mr - nt)\sin^2\theta \\ a &= \frac{n\Phi}{r^2}\cos(mr - nt)\sin\theta \end{aligned}\right\} \quad . \; . \; . \quad (12)$$

It is to be noticed that this expression for a is that which, according to the formula at p. 143, would be given for the magnetic force produced by a current γ in an element of length ds such that $\gamma ds = n\Phi\cos(mr - nt)$. But $n\Phi\cos(mr - nt)$ is the actual current in the doublet at any instant multiplied by the length of the element. The theory therefore leads to the law there stated.

Forces at Points Distant from the Vibrator.

At a great distance from the origin the equations (11) for the components become

$$\left.\begin{aligned} KQ &= -\frac{\Phi}{r}m^2\sin(mr - nt)\sin\theta\cos\theta \\ KR &= \frac{\Phi}{r}m^2\sin(mr - nt)\sin^2\theta \\ a &= \frac{n\Phi}{r}m\sin(mr - nt)\sin\theta \end{aligned}\right\} \quad . \; . \quad (13)$$

At Distant Points Electric and Magnetic Force Propagated Together in Transverse Vibrations.

These equations show that at great distances from the origin the electric and magnetic forces are propagated together, with the velocity n/m, and are in the same phase.

Further the first two equations of (18) show that when r is great $Q\sin\theta + R\cos\theta = 0$, which indicates that the direction of vibration is perpendicular to the radius vector from each point, that is that the vibrations are *transverse* to the direction of propagation.

Along the axis of z, $a = 0$, $Q = 0$, and the equation for R in (11) may be written

$$KR = \frac{2\Phi}{r^3}\sqrt{1 + m^2r^2}\sin(mr - nt - e) \quad . \; . \; . \quad (14)$$

where $\tan e = mr$.

Propagation along the Axis of z and in the Equatorial Plane:

Thus the velocity of propagation of electric force along the axis of z is to be found from the equation $mr - nt - e = 0$ by calculating dr/dt, and is therefore $n(1 + m^2r^2)/m^3r^2$. It is very great when r is small, and approaches the value n/m, or $1/\sqrt{K\mu}$ as r increases.

In the equatorial plane $Q = 0$, and the equations for R and α may be written

$$\left.\begin{aligned} KR &= \frac{\Phi}{r^3}\sqrt{1 - m^2r^2 + m^4r^4}\,\sin(mr - nt - e) \\ \alpha &= \frac{n\Phi}{r^2}\sqrt{1 + m^2r^2}\,\sin(mr - nt - e') \end{aligned}\right\} \quad . \quad (15)$$

where $\tan e = mr/(1 - m^2r^2)$, $\tan e' = -1/mr$.

Velocity about Origin and at Great Distance.

The velocity of propagation of electric force in the equatorial plane is thus $n(m^4r^4 - m^2r^2 + 1)/m^3r^2(m^2r^2 - 2)$. It is always greater than the velocity along the axis, except of course for great values of r, where it is n/m as in the other case. Moreover it is infinite when $r = 0$, and when $r^2 = 2/m^2$, or $r = \lambda/(\pi\sqrt{2})$, and is negative at intermediate points. The electric force is thus propagated outwards and inwards in the equatorial plane from a point outside the vibrator. This is the point for which $r = \lambda/(\pi\sqrt{2})$, the centre of the small circle seen on each side of the vibrator in the graphic representation of the electric field of the vibrator in Fig. 181 below. At this point the electric force attains any value which it there takes ·12 of a period before the corresponding value of the force is attained at the origin.

Velocity of Propagation of Magnetic Force.

The velocity of propagation of the magnetic force in the equatorial plane is $n(1 + m^2r^2)/m^3r^2$, which is also infinite at the origin but diminishes as r is increased towards the limiting value n/m.

Acceleration of Phase of Electric and Magnetic Forces.

It is easy to verify from the expressions here given * that the interval in which a zero or maximum value of the magnetic force travels out in the equatorial plane from the origin to a great distance r is $rm/n - T/4$, and that a zero value of the electric force travelling out in the same plane from the point $\lambda/(\pi\sqrt{2})$ reaches the point r in the interval $rm/n - T/2$ after the instant at which the zero value reached the origin. But when this value reaches the origin the current has its maximum value, and therefore so has the magnetic force.

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* See also Trouton, "On the Acceleration of Secondary Electromagnetic Waves," *Phil. Mag.* March, 1890.

In the succeding interval $rm/n - T/4$ this maximum travels out a distance r, but the zero value of the electric force has arrived earlier by T/4, so that the electric force is also a maximum at the same time as the magnetic force, in accordance with the equations (18) below.

Lines of Electric Force.

Lines of electric force have the differential equation

$$\frac{dz}{R} = \frac{d\rho}{Q},$$

or

$$\frac{dz}{-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Pi}{\partial\rho}\right)} = \frac{d\rho}{\frac{\partial^2\Pi}{\partial\rho\partial z}},$$

that is

$$\frac{\partial}{\partial z}\left(\rho\frac{\partial\Pi}{\partial\rho}\right)dz + \frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Pi}{\partial\rho}\right)d\rho = 0. \quad . \quad . \quad . \quad (16)$$

Integrating this we get for the equation of a line of force $\rho\partial\Pi/\partial\rho = c$, or

$$\frac{\Phi}{r}\{\sin(mr - nt) - mr\cos(mr - nt)\}\sin^2\theta = c \quad . \quad (17)$$

where c is a constant for any particular line. By assigning different values to c the whole family of lines of force existing at any particular instant can be obtained.

Graphic Representation of Change of Field during Half-Period.

The curves of electric force as plotted by Hertz are reproduced in Figs. 180, 181, 182, 183. Fig. 180 shows the electric field as it exists at the beginning of an oscillation when the vibrator is in the neutral state, Figs. 181, 182, 183 after the lapse of successive eighths of a complete period. The figures were drawn by means of a set of values of $\sin^2\theta$, and corresponding values of θ, and an auxiliary curve giving the values of r for which (with the given values of t) the multiplier of $\sin^2\theta$ in (17) gives a product equal to the constant c chosen for the curve. [The λ on the curves is $\frac{1}{2}$ of the wave-length.]

The lines in the immediate vicinity of the vibration are not given, nor are those drawn continued up to the source. The vibrator represented is of a dumb-bell shape, and therefore its lines of force can only agree with those of a doublet at some distance from the origin.

Fig. 181 gives the state of the field after the lapse of $\frac{1}{8}$ of a period from the instant for which Fig. 180 is drawn. The lines shown are enclosed within the circle given by (17) for $t = \frac{1}{8}T$,

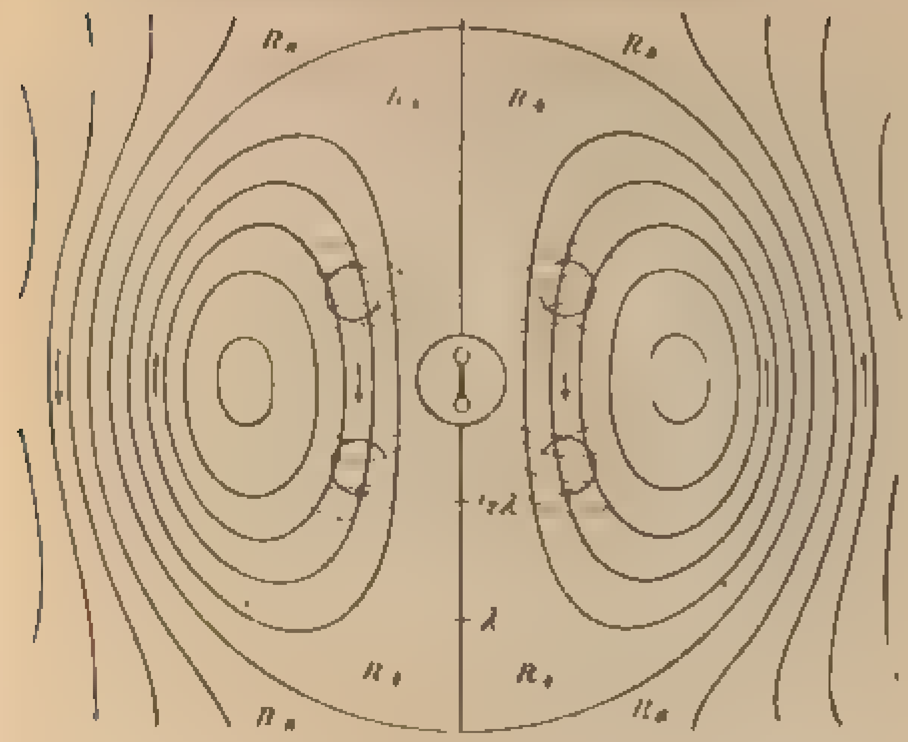


FIG. 180

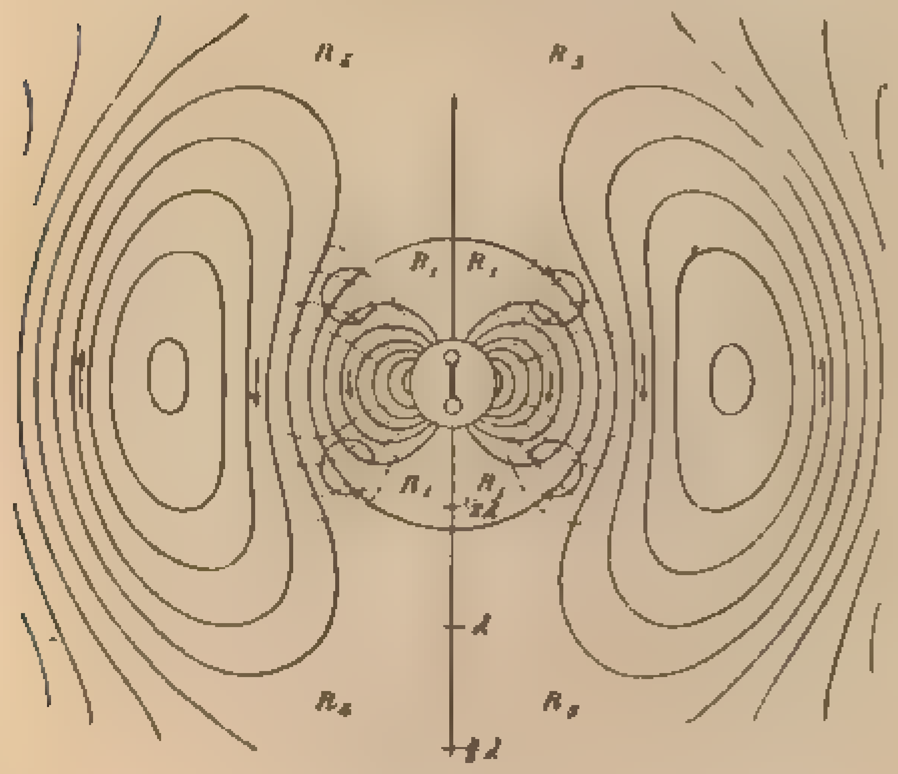


FIG. 181

and $c=0$. This circle travels outwards with the same velocity as the magnetic force. [The closed curves outside the circle may for the present be neglected.]

In Fig. 182, which shows the field when $t=\frac{3}{8}T$, these lines and enclosing circle are seen to have spread out to a greater distance. In Fig. 183 the enclosing circle is much larger, but in the immediate neighbourhood of the vibrator the lines of force have begun to contract inwards on the source. The curves are thus throttled, so to speak, and break off at the neck into closed curves. These closed curves are formed first in the interior of the system as shown at the small circles to right and left of the vibrator inside the outer loop of the dotted curve. At these points, as already stated, the electric force takes any possible value before the corresponding value is reached at the origin.

Motion of Line of Force in Field. For example in Fig. 183 the electric force has just become zero at the small circles. As t increases from $\frac{3}{8}T$ to $\frac{1}{2}T$ the curves break off successively from within outwards until, as shown in Fig. 180, they have all broken off into two groups of closed curves seen to right and left of the origin within the circle. These are as it were the cross-sections of a vortex which is produced and remains symmetrically round the axis of the vibrator. Its circular axis at the small circles increases in radius at first very rapidly but ultimately with the speed of light. [The arrows in the curves in Fig. 183 are opposed in direction to those in Fig. 180, but this arises from the fact that the lines in Fig. 180 are really those formed in the half-period preceding that now under consideration.]

Considering now the interval from $t=\frac{1}{2}T$ to $t=\frac{5}{8}T$ we see that lines begin to spread out just as before, except that the directions of the forces are reversed. These lines force outwards the closed curves of the "vortex" just thrown off, rendering them concave on their inner sides and more and more elongated, as shown in the successive diagrams, so that they approximate more and more to lines transverse to the radius-vector drawn from the origin to any point.

Radiation of Energy. With the breaking off and motion outwards of these curves is connected the radiation of energy. Progress outwards of the state of stress indicated by these lines involves the carrying off into surrounding space of part of the energy supplied by the medium to start the vibrator, the oscillations of which are therefore subject to a damping action over and above that due to the resistance of the bar or medium connecting the conductors.

The rate of radiation of energy can be calculated by Poynting's theorem. By the formula given at p. 214 above (which may be proved independently for the present case by multiplying

[redacted]

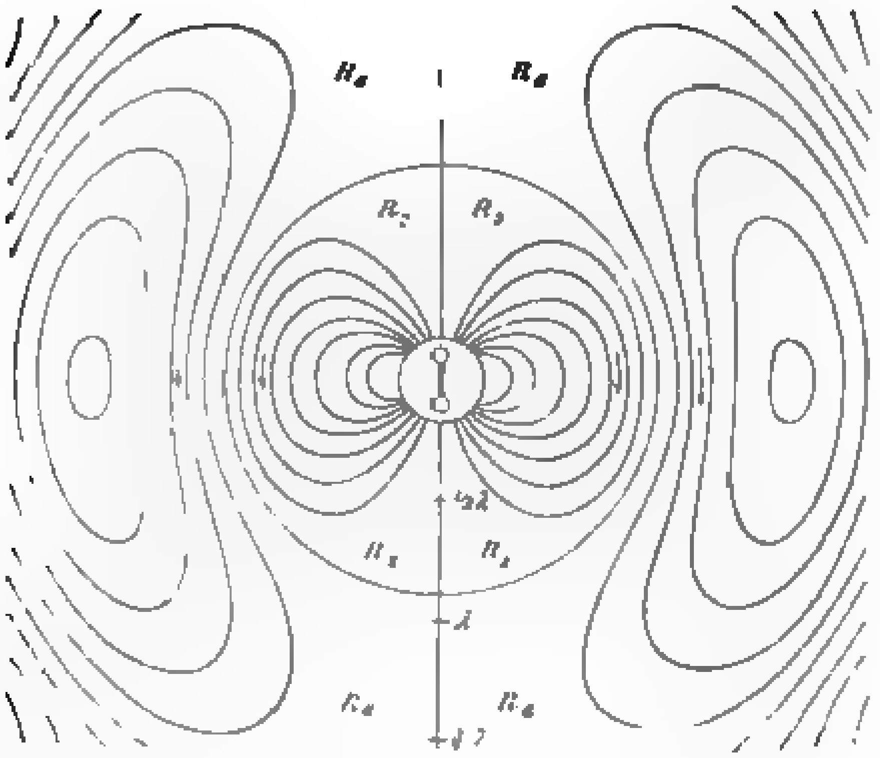


Fig. 182.

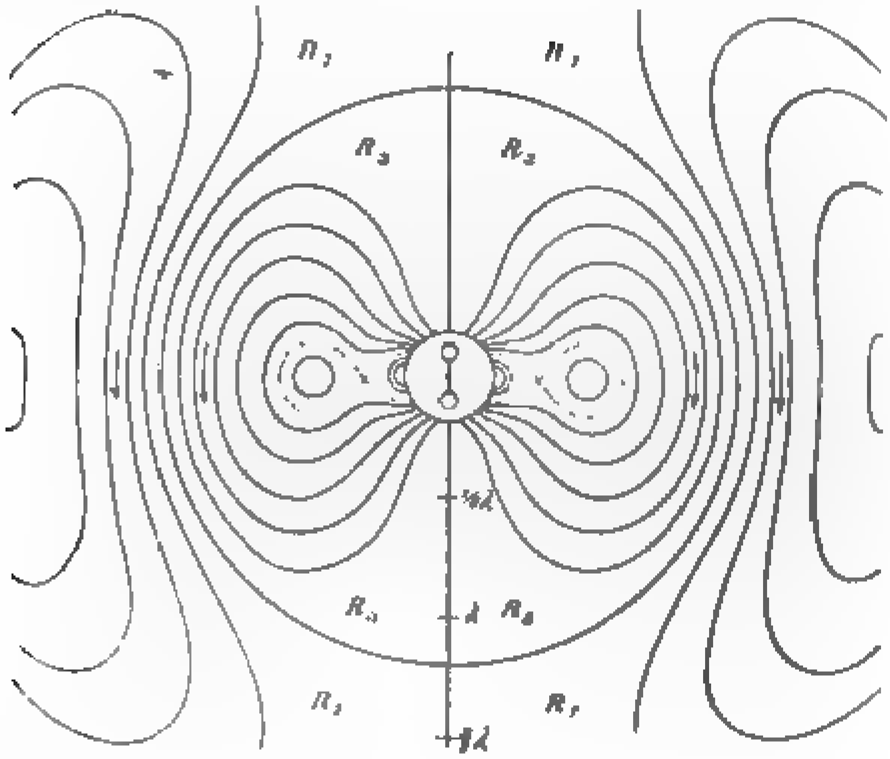


Fig. 183

Calculation of Rate of Radiation.

equations (1) by P, Q, R, (2) by α, β, γ, respectively, subtracting the sum of the second set of products from the sum of the first set, and integrating throughout the space within the closed surface chosen), the rate of flow of electric and magnetic energy combined across unit area of surface at any point is

$$\mathbf{E}\mathbf{H} \sin \phi \sin \psi / 4\pi$$

where **E**, **H**, are the resultant electric and magnetic forces at the point, ϕ the angle between their directions, and ψ the angle between the outward normal to the surface at the point and the common normal to **E** and **H**.

It will be sufficient to calculate for a sphere of large radius described from the origin as centre. In this case we have by (13)

$$\left.\begin{aligned} K\mathbf{E} &= \sqrt{Q^2 + R^2} = \frac{\Phi}{r} m^2 \sin(mr - nt) \sin\theta \\ \mathbf{H} &= \alpha \qquad\quad = \frac{\Phi}{r} mn \sin(mr - nt) \sin\theta \end{aligned}\right\} \quad . \quad . \quad . \quad (18)$$

E and **H** are at right angles to one another and lie in the tangent plane to the sphere at every point. The rate of flow of energy per unit of area across the surface is therefore $\Phi^2 m^3 n \sin^2(mr - nt) \sin^2\theta/(4\pi K r^2)$, and by the rule given at p. 215 the direction of flow is outwards.

The total rate of flow of energy across a zone of the spherical surface of angular distance θ from the axis and breadth $r d\theta$ is thus $\Phi^2 m^3 n \sin^2(mr - nt) \sin^3\theta d\theta / 2K$. Integrating this from 0 to π we find for the total rate of radiation across the surface

$$\frac{2}{3} \frac{\Phi^2}{K} m^3 n \sin^2(mr - nt).$$

Again, integrating this expression over half a period, say from $t = 0$ to $t = T/2$, we get finally for the energy radiated in each half-period the value

$$\frac{1}{6} \frac{\Phi^2}{K} m^3 n T = \frac{\pi}{3} \frac{\Phi^2}{K} m^3 = \frac{\Phi^2}{3K} \frac{8\pi}{\lambda^3} \quad . \quad . \quad . \quad (19)$$

If we measure Φ in electrostatic units we must take K in the same units. Then for air $K = 1$. Before obtaining numerical results we must calculate the period of the vibrator.

Calculation of Period of Vibrator.

The period of the vibrator, taken as $2\pi\sqrt{LC}$ (p. 188) by neglecting R, can be found by calculating L by Neumann's formula [(47) p. 171], and taking account of the displacement currents in the dielectric. If a current element produces the magnetic field given by the formula of p. 143, and the displacement current be uniformly and radially distributed in the dielectric round each end of the wire,* the correction for a pair of elements ds_1, ds_2, at distance r is $\frac{1}{2}\,d^2r/ds_1ds_2 . ds_1ds_2$, and hence for any two linear currents the correction is the integral of this taken along both lines. Thus if P_1, P_2, be the ends at which the currents enter the wires, N_1, N_2 those by which they leave, the integral is obviously $\frac{1}{2}(P_1P_2 + N_1N_2 - P_1N_2 - P_2N_1)$.

Hence, as the reader may verify, if we consider a straight conductor of length l and take two parallel filaments of it at distance x apart and integrate along both, we get

$$\iint\left(\frac{\cos\theta}{r} + \frac{1}{2}\frac{d^2r}{ds_1ds_2}\right)ds_1ds_2 = 2l\log\frac{l+\sqrt{l^2+x^2}}{x} + 3(x - \sqrt{l^2+x^2})$$

$$= 2l\left(\log\frac{2l}{x} - \frac{3}{2}\right)\dagger \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

if x/l may be neglected. We may extend this to all the filaments of the conductor by taking instead of x the geometric mean distance (p. 290) of the current carrying section from itself. If we suppose the current only on the surface of the conductor the G.M.D. is simply the radius a and we get

$$L = 2\mu l\left(\log\frac{2l}{a} - \frac{3}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

If the current is taken as uniform over the cross-section log (G.M.D.) is (p. 296) log $a - \frac{1}{4}$, so that we have to substitute $\frac{5}{4}$ instead of the $\frac{3}{2}$ in the above result. Taking μ as unity, l as 100 cms., a as ·25 cm., as in Hertz's dumb-bell apparatus, we get $L = 1037$ by (21). Since the spheres were 15 cms in radius we must take $C = 7{\cdot}5/v^2$, where v is the ratio of the units (Chap. XI.). Thus we get $T = 1{\cdot}85 \times 10^{-8}$ of a second.

* See Heaviside, *Electrician*, Dec. 28, 1888

† Hertz calculating by a formula given by Helmholtz, differing from the above only in having d^2r/ds_1ds_2 multiplied by $(1-k)/2$, where k is an undetermined constant, finds $L = 2l\,[\log(2l/a) - 3/4 + (1-k)/2]$, the current being supposed uniform over the cross-section. The last term should be $(k-1)/2$. From this by putting $k=0$, we get the result given above.

Numerical Value of Rate of Radiation.

In experiments made by Hertz the conductors were two equal spheres, 15 cms. in radius, placed with their centres 100 cms. apart. These were charged to a difference of potential which gave a spark-distance of 1 cm. The potential of each differed from zero therefore by about 60 C.G.S. electrostatic units, and the charge of each sphere was 60×15 C.G.S. units. Thus $\Phi = 60 \times 15 \times 100$, and the energy radiated in half a period was $60^2 \times 15^2 \times 100^2 \times 8\pi^4/3\lambda^3$ ergs, λ being taken in cms. If the velocity was that of light the wave-length was about 550 cms., and hence in each half-period about 12000 ergs passed from the vibrator into the surrounding medium.

The whole energy of the vibrator when charged to the potential stated above was $\frac{1}{2} \times 2 \times 60^2 \times 15$ $(=54000)$ ergs. Thus about $\frac{2}{9}$ of the whole initial energy was radiated in the first half-period, that is the amplitude of vibration suffered from radiation alone a diminution of roughly $\frac{1}{9}$.

The period being $1{\cdot}85 \times 10^{-8}$ second the rate of radiation of energy was therefore about $1{\cdot}34 \times 10^{12}$ ergs per second, or approximately $1{\cdot}34 \times 10^4/746$ $(=179)$ horse-power.

Propagation of Waves along Wires.

Experiments were made by Hertz, as described below, on the interference of waves propagated along wires. The successive oppositions of phase were found to occur at much smaller intervals in the neighbourhood of the vibrator than at a great distance. It was at first thought that this result was due to the existence of two effects, an electrostatic one travelling at an infinite speed, and an electromagnetic one travelling with the velocity of light. The theory given above shows that this is not the true view of the case, and we must examine what phenomena of interference with waves in wires are given by it. We shall consider first the propagation of waves in wires.

Theory of Waves in Wires.

To find a solution of this problem we shall suppose the wire to lie along a new axis of z, and put for the value of Π at any point of the wire the expression

$$\Pi = A \sin (m'z' - n't + \delta) \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where n'/m' denotes the speed of transmission, whatever it is, of a given value of Π along the wire, z' the distance of the point considered from a point chosen as origin, and δ the phase angle of the vibration at the origin when $t=o$. The wire in the interference experiments referred to was placed horizontally in the plane of symmetry of the vibrator, but 40 cms. higher.

If we suppose no damping out of the wave or change of form to take place A cannot be a function of z or t, and is

therefore a function of ρ. We write therefore instead of (22),

$$\Pi = f(\rho)\sin(m'z' - n't + \delta) \quad . \quad . \quad . \quad . \quad . \quad (23)$$

But for any point in the insulating medium surrounding the wire we have the differential equation

$$\frac{\partial^2 \Pi}{\partial t^2} = \frac{1}{K\mu}\nabla^2 \Pi \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

By substitution from (23) in (24) we find

Differential Equation of Condition.

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} - (m'^2 - n'^2 K\mu)f = 0 \quad . \quad . \quad . \quad . \quad (25)$$

Solution.

This equation may be solved in the following manner. Imagine a linear distribution of attracting matter along the axis of z', such that between two points at distances $z + \zeta$ and $z' + \zeta + d\zeta$ from the origin the quantity of matter is $\cos p(z' + \zeta)d\zeta$. The

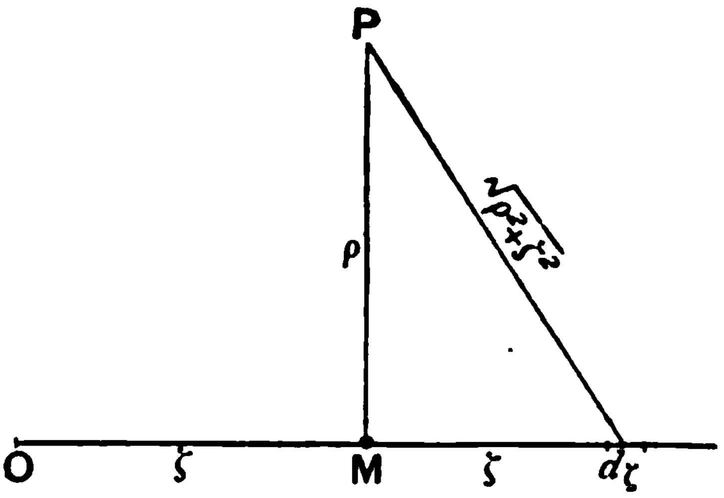


FIG. 184.

potential at a point P at distance r from the origin due to the element $d\zeta$ is $\cos p(z' + \zeta)d\zeta/r$. We shall take z as the axial coordinate, OM, of the point P, Fig. 184. If matter be distributed according to this law to an infinite distance on both sides of the origin the potential will be given by the equation

$$V = \int_{-\infty}^{+\infty} \frac{\cos p(z' + \zeta)}{r}\, d\zeta = \int_{-\infty}^{+\infty} \frac{\cos p(z' + \zeta)}{\sqrt{\rho^2 + \zeta^2}}\, d\zeta$$

$$= \cos pz' \int_{-\infty}^{+\infty} \frac{\cos p\zeta}{\sqrt{\rho^2 + \zeta^2}}\, d\zeta \quad . \quad . \quad . \quad . \quad . \quad (26)$$

since

$$\int_{-\infty}^{+\infty} \frac{\sin p\zeta}{\sqrt{\rho^2+\zeta^2}} d\zeta = 0,$$

each element for a positive value of ζ being cancelled by a corresponding element for ζ negative.

The integral in (26) does not depend in any way upon z or ζ, and is therefore entirely a function of ρ. But at every point P the potential V fulfils Laplace's equation, which in the present case has the form

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} - p^2 V = 0.$$

If we write V in the form $\phi(\rho)\cos pz$ we obtain from the last equation

$$\frac{\partial^2 \phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \phi}{\partial \rho} - p^2 \phi = 0,$$

which is the equation fulfilled by $f(\rho)$ if

$$p^2 = m'^2 - n'^2 K\mu.$$

Hence we may put

$$f(\rho) = \frac{2J}{n'}\int_{-\infty}^{+\infty} \frac{\cos\sqrt{m'^2 - n'^2 K\mu}\,.\,\zeta}{\sqrt{\rho^2+\zeta^2}} d\zeta \quad . \quad . \quad . \quad . \qquad (27)$$

where J is a constant, and finally *

$$\Pi = \frac{2J}{n'}\sin(m'z' - n't + \delta)\int_{-\infty}^{+\infty} \frac{\cos\sqrt{m'^2 - n'^2 K\mu}\,\zeta}{\sqrt{\rho^2+\zeta^2}} d\zeta\,. \qquad (28)$$

This is the general solution from which the electric and magnetic forces are to be found by (4).

* The solution here adopted is that given by Poincaré (*Electricité et Optique*, tome ii. p. 192). Hertz gives practically the same solution but in a somewhat different form. He remarks that the integral

$$\int_0^\infty \epsilon^{-\frac{1}{2}p\rho\left(\epsilon^u+\epsilon^{-u}\right)} du$$

satisfies the differential equation (25). If we assume that p is not less than zero, that is, that the velocity of propagation along the wire is

[redacted]

If we now impose the condition that the electric forces are at right angles to the conducting wire we must have $R = 0$, that is by (7)

If Electric Forces Normal to Wire Velocity of Propagation = Velocity of Light.

$$\frac{\partial^2 \Pi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Pi}{\partial \rho} = 0,$$

or

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

This gives $f = C \log \rho + C'$, where C, C', are constants, and therefore we may write

$$\Pi = \frac{2J}{n'} \log \rho \, . \sin (m'z' - n't + \delta) \quad . \quad . \quad . \quad . \quad (30)$$

---

not greater than in the surrounding medium, we may write $p\rho \equiv -i\theta$, where θ is a pure positive imaginary, and $i = \sqrt{-1}$. The integral then becomes

$$\int_0^\infty \epsilon^{i\theta \cos iu} \, du$$

and is a case of the function $K(\theta)$ discussed in treatises on Bessel's functions.

Now it is proved (Heine's *Kugelfunctionen*, p. 188, 2nd edition) that if θ be as here a pure positive imaginary

$$K(\theta) = \int_0^\infty \cos (\theta \sin iu) \, du = \int_0^\infty \frac{\cos p\rho x}{\sqrt{1 + x^2}} \, dx$$

$$= \int_0^\infty \frac{\cos p\zeta}{\sqrt{\rho^2 + \zeta^2}} \, d\zeta$$

if for ρx we write ζ. But

$$\int_{-\infty}^{+\infty} \frac{\cos p\zeta}{\sqrt{\rho^2 + \zeta^2}} \, d\zeta = 2 \int_0^\infty \frac{\cos p\zeta}{\sqrt{\rho^2 + \zeta^2}} \, d\zeta.$$

Thus the two solutions agree to a constant multiplier.

[redacted]

By comparison of (25) and (31) we see that in this case

$$p^2 = m'^2 - n'^2 K\mu = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

that is the velocity of propagation of the wave along the wire is equal to the velocity of light.

Lines of Force for Different Velocities of Propagation.

Fig. 185 shows the lines of electric force in the neighbourhood of the wire for $p = m^2$, that is for an infinitely small velocity of propagation along the wire. Here the lines make finite angles with the wire except at pairs of points at successive distances of half a wave length, along the wire, where R is zero. The field is wholly electrostatic. Fig. 186 shows the change produced in the field when the velocity of propagation along the wire is 7/12 of that in the medium. The component of electric force along the wire is weakened by electrodynamic action, and the curves run farther out since the tangent at any point is thus rendered more nearly perpendicular to the wire.

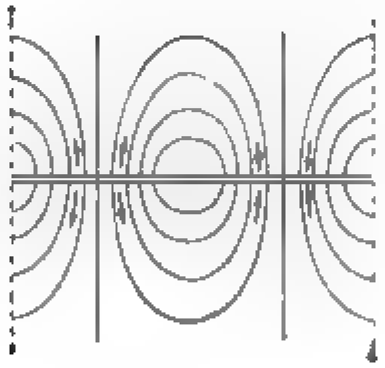


FIG. 185.

In Fig. 187 the field in the case of velocity of transmission equal to that of light is shown. The lines of electric force are everywhere in planes at right angles to the axis; each of the curves of the former diagram has by the vanishing of R at every point been changed into a pair of parallel lines, as indicated by the arrows, one running out to infinity the other returning from infinity to the wire.

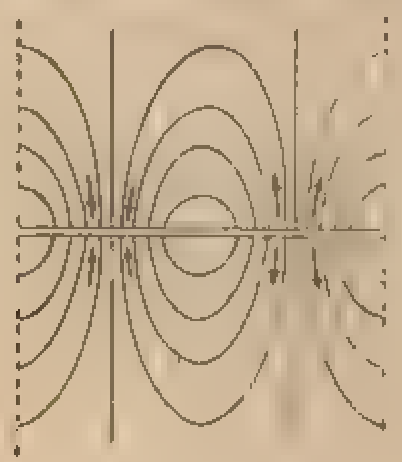
Taking now the component of electric force in the equatorial plane as given for the vibrating doublet by (15), we may write it

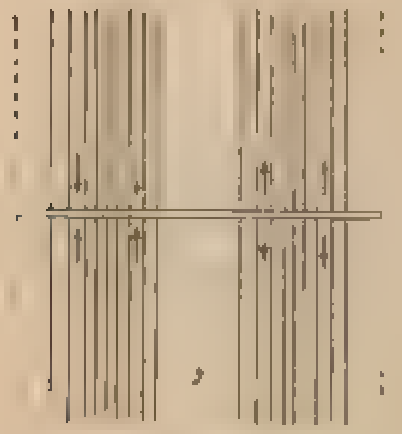
$$KR = -\frac{\Phi}{r^3}\sqrt{1 - m^2r^2 + m^4r^4}\,\sin(nt - \delta_1) = C\sin(nt - \delta_1) \quad (32)$$

where

$$\delta_1 = mr - \tan^{-1}\frac{mr}{1 - m^2r^2}.$$

Plotting δ_1 as ordinates of a curve with values of mr as the abscissæ, Hertz obtained the curve marked δ_1 in Fig. 188.* This





FIGS. 186, 187.

curve is drawn for the experiments described at p. 812 below. The scale below indicates metres, and starts from a point (45 cms. to the right of the origin of abscissæ) at which the value

* This cut is copied from Hertz's paper, "Die Kräfte elektrischer Schwingungen," *Wied. Ann.* 36 (1888), p. 1. See also *Nature*, vol. 39 (1888—9), p. 547.

of δ_1 begins to be sensible. The distance ab along the axes of x and y represents π.

Interferences of Direct and Wire Waves as given by Theory.

It will be seen from the curve that it is asymptotic to the straight line ($\delta = mr - \pi$) drawn from b, and that therefore at a great distance the phase alters uniformly as if the wave had been transmitted over the first half wave-length in an infinitely short time, and had thereafter travelled with velocity n/m.

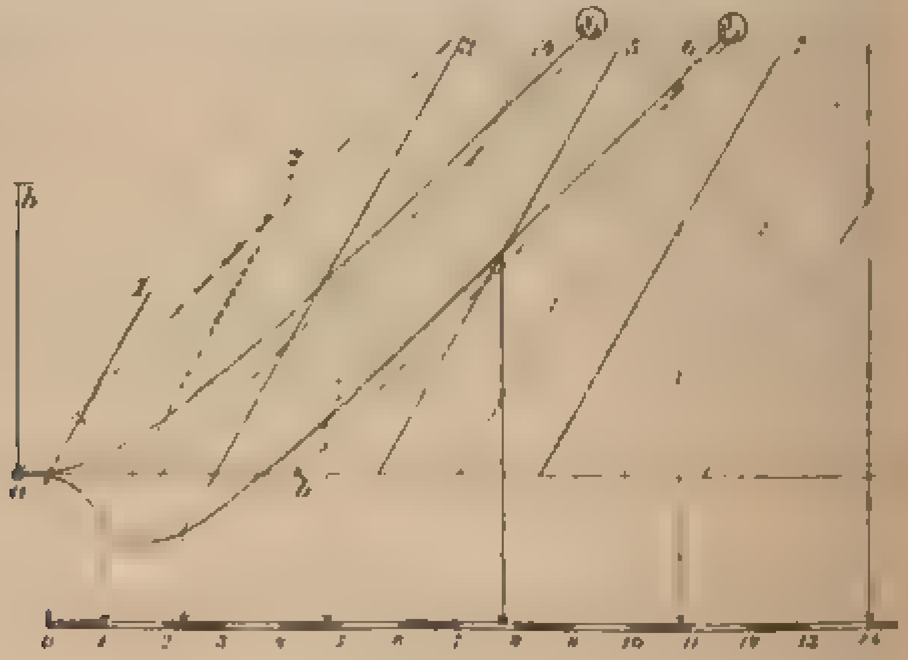


FIG. 188.

The electric force parallel to z' at any point in a plane through the wire, and produced by the disturbance in the wire, has the expression

$$W = C \sin (n't - \delta_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where by (30)

$$\delta_2 = m'z' + \delta = \frac{2\pi}{\lambda'} z' + \delta,$$

if λ' denote the length of the waves in the wire. The value of δ and the amplitude C can be altered to any required amount by properly adjusting the length of the wire.

Considering then the interference between the electric force in the equatorial plane due to the radiation from the vibrator, and that produced by the wire, we see that the phase of the

resultant depends on $\delta_1 - \delta_2$. For if the resultant be R' we have, if $C' = C$,

$$R' = R + W = C\{\sin(nt - \delta_1) + \sin(n't - \delta_2)\}$$

$$= 2C \sin\left(\frac{n + n'}{2}t - \frac{\delta_1 + \delta_2}{2}\right)\cos\frac{\delta_1 - \delta_2}{2} \quad . \quad . \quad (34)$$

Thus if $\delta_1 - \delta_2$ is zero or a multiple of 2π, $\cos\{(\delta_1 - \delta_2)/2\} = \pm 1$ and the effects conspire. If $\delta_1 - \delta_2$ is an odd multiple of π $\cos\{(\delta_1 - \delta_2)/2\} = 0$, and the effects are opposed. In the former case the interference may be called $+$, in the latter $-$, while in the case of $\delta_1 - \delta_2$, an odd multiple of $\pi/2$, it may be said to be zero.

Now suppose that at the zero of the metre division $\delta_1 - \delta_2$ has some definite value δ_0, and let a straight line (1 in Fig. 188) be drawn to represent $\delta_2 + \delta_0$. Its slope must be such that it rises π for a distance along the axis of abscissæ equal to half a wave-length in the wire. In the figure it is thus drawn for a wave-length of 5·6 metres, to suit the experiments made by Hertz. The lines numbered 2, 3, 4, &c., are drawn in the same way to represent $\delta_2 + \delta_0 - \frac{1}{2}\pi$, $\delta_2 + \delta_0 - \pi$, $\delta_2 + \delta_0 - \frac{3}{2}\pi$, &c. These cut the axis of abscissæ at successive distances from the origin of the metre scale of 1·4 metres, and are all parallel to the line numbered 1.

The intersections of these lines with the curve δ_1 projected on the axis of abscissæ give the distances from the origin at which δ_1 has the successive values $\delta_2 + \delta_0$, $\delta_2 + \delta_0 - \frac{1}{2}\pi$, &c. Thus if the interference at the beginning of line 1 have the sign $+$ $(-)$, it will have $-$ $(+)$ at a distance of 2·3 metres, $+$ $(-)$ at a distance of 7·6 metres, $-$ $(+)$ at 14 metres, and so on. Or if it be zero at the beginning of the metre scale, it will be zero at distances 2·3, 7·6, 14, &c. metres, and have opposite signs at the intermediate distances 1 metre, 4·8 metres, 11 metres, &c.

This it will be seen below expresses the experimental results. The retardation of the magnetic force given by the equation $\delta_3 = m_3 \tan^{-1} mr$ is shown by the line δ_3. Further discussion of the experimental results, including the interference of the magnetic actions, will however come more conveniently after a description of the experiments made by Hertz and others.

Some of the most important of Hertz's experiments were carried out by means of the dumb-bell vibrator and receiver referred to above, or with the dumb-bell vibrator modified by substituting for the spheres plates coplanar with the axis. The vibrator, Hertz's Experiments.

Mode of Production of Oscillations.

which was placed horizontally, was connected as shown in Fig. 189 to the terminals of an induction coil. The two spheres A, A' were thus charged to a difference of potential sufficient to enable a spark to pass across the gap. As soon as the spark passed electric oscillations were set up which depended for their period on the dimensions of the apparatus, but were enormously more rapid than the action of the coil. Thus the oscillations had subsided in consequence of generation of heat and radiation of energy long before they were renewed by the coil. The action of the vibrator was therefore a succession of oscillatory discharges separated by intervals of inaction.

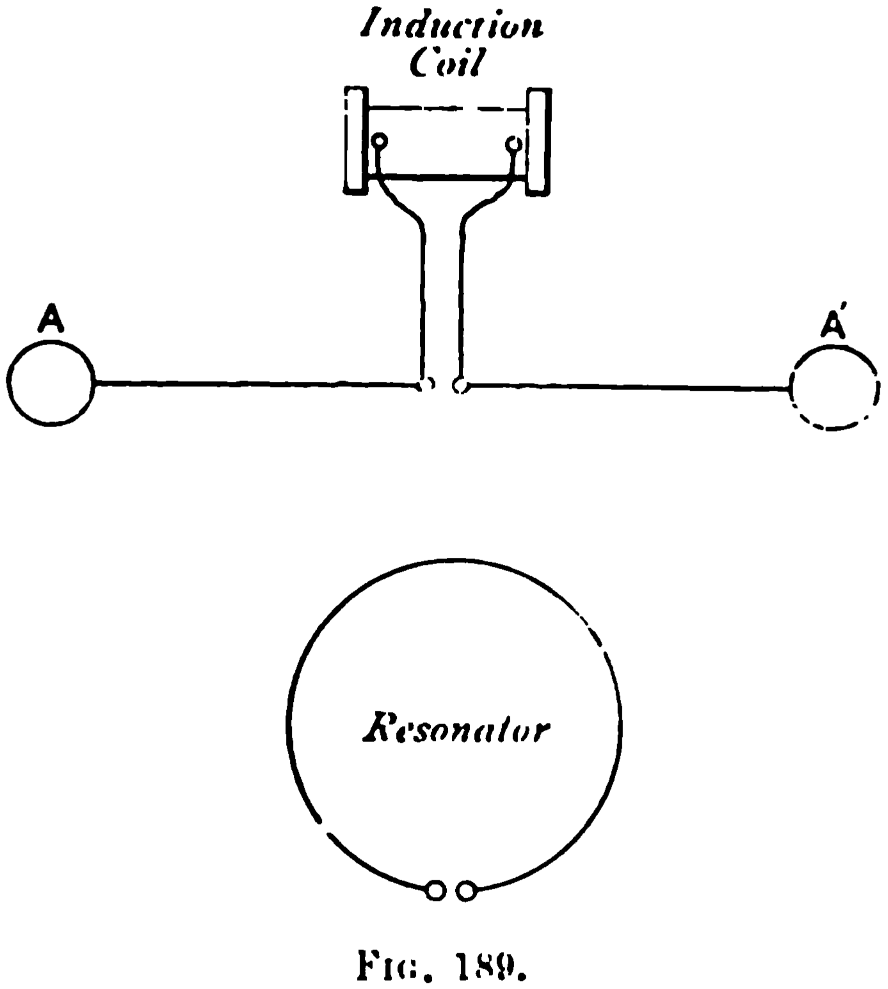


Fig. 189.

Resonating Action of Receiver.

What was very remarkable in the receiver was the fact that a particular size depending mainly on the vibrator was necessary to enable it to act with greatest readiness and intensity. This suggested that its behaviour was that of an electric resonator, so to speak, tuned to respond to the vibrations which it received, and to no others. Of course this was only approximately the case: the electric radiation was essentially a compound phenomenon, and indeed very accurate tuning has been found unnecessary for obtaining results.

The wire of which the receiver was made was 2 mms. thick and the diameter of the circle was 35 cms. The ends of the wire at the spark-gap were tipped with two small knobs. The width of the gap was regulated by means of a fine screw, which moved one end of the wire. Dimensions of Receiver.

In some experiments with the same vibrator a receiver in the form of a square 60 cms. in side, made of wire of the same gauge, was used. The spark-gap was in the middle of one of the sides.

It is impossible to give a complete account of the action of the receiver or resonator, but some approach to a rough theory can be made which serves, partly at least, to explain the results obtained. [See Hertz, *Wied. Ann.* 34, p. 155 (1888), or, *Untersuch*, p. 89.] Theory of Receiver.

If we denote by P the electric force, parallel to an element ds of the resonating circle, produced by the action of the exciter, we shall have

$$P = \phi(s) \cos nt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

on the supposition that the electric force is oscillatory with period $2\pi/n$, and is a function of the distance s of the element from some point of the circle (say the centre of the spark-gap) taken as origin. We assume further that $\phi(s)$ is a periodic function of s, which is no doubt the case when electrical oscillations are going on in the circle. Hence by Fourier's series we get

$$\phi(s) = A + B \cos \frac{2\pi}{S} s + \ldots + B' \sin \frac{2\pi}{S} s + \ldots \quad (36)$$

Electric Force at Point on Receiver

where S denotes the whole circumference of the circle. It is not necessary to retain more than the three terms here exhibited, the constant term and the two indicating the gravest simple harmonic component. If we take the origin at the spark-gap, we may neglect also the term $B' \sin (2\pi s/S)$ since that is of opposite sign in the two halves of the circle on the two sides of the diameter through the spark-gap.

At the spark-gap we have then by (36)

$$\phi(s) = A + B$$

and at the diametrically opposite point

$$\phi(s) = A - B$$

The current at the spark-gap is of course the rate of passage of electricity in the spark itself.

Action of Electric Force on Resonator.

According to Hertz's view the action of the vibrator on the portion of the resonator opposite the spark-gap was most effective in setting up the oscillation, that is, that the oscillations depend rather on $A - B$ than on $A + B$. Such an oscillation no doubt consists first in a backward and forward flow of electricity in the connecting wire from one knob to the other, which gradually increases in amplitude, if the receiver is of natural period nearly equal to that of the vibrator, until the maximum difference of potential between the knobs becomes so great that a spark passes.

Forced Oscillations in Resonator.

He compares the resonator to a string fixed at its two ends and subjected to periodic forces at intermediate points, so that it is set into a state of forced vibration. Thus if V denote the difference of potential of the knobs at any time, we have, on the supposition that the exciting vibrations are slowly damped, the excited not,

$$\frac{\partial^2 V}{\partial t^2} + p^2 V = A' e^{-\kappa t} \sin nt \quad . \quad . \quad . \qquad (37)$$

if $2\pi/p$ be the natural period of free vibration. The solution for forced vibration is

$$V = \frac{A' e^{-\kappa t}}{\sqrt{(n^2 - p^2 - \kappa^2)^2 + 4n^2\kappa^2}} \sin (nt - e) + B \sin pt + C \cos pt. \qquad (38)$$

where $\tan e = 2n\kappa / (n^2 - p^2 - \kappa^2)$.

If κ be supposed very small, V will be very great if p is equal to n, and will have the same sign as A' or the opposite sign, that is the phases of the forced part of V and the exciting action will agree or be opposed, according as $p >$ or $< n$. But if $p = n$, $\tan e$ is very great and approximately $e = \pi/2$. Thus when resonance is just attained there is a difference of phase of a quarter of a period. If κ is not very small, the forced vibration is a maximum when $n^2 = p^2 + \kappa^2$, and the greater κ is the smaller is this maximum.

Electric Forces on Element of Resonator.

The terms of P may be interpreted as follows: $A \cos nt$ is an electric force at the element ds, which has the same value for each element at the same instant. It may be taken as the electric force due to variation of magnetic induction of the same value at every point of the circle, or if the magnetic induction is not uniform, the part of this electromotive force which is the same for each element. In the second term is included the so-called electrostatic action of the vibrator, and any remaining portion of inductive action however produced.

If E be the part of the electric force which is independent of that due to the variation of the uniform part of the magnetic

induction, ψ the angle which it makes with the plane of the circle, and θ the angle between the component in the plane of the circle and the radius to the centre of the spark-gap, the component along the tangent is $E \cos\psi \sin(2\pi s/S - \theta)$. Therefore

$$R = -E \cos\psi \sin\theta \cos nt.$$

Quantity on which Length of Spark depends. Experiments with Resonator Vertical.

Hence the length of spark may be taken as roughly depending on the value of a quantity of the form $\alpha + \beta \sin\theta$, where α is proportional to A and β to $E\cos\psi$.

In the experiments *loc. cit.* it was of course sufficient, on account of the symmetry of the arrangement, to investigate what took place for positions of the centre of the receiver at different points of one of the four quadrants into which the horizontal plane through the vibrator was divided by the line of the vibrator itself, and the horizontal perpendicular to the vibrator passing through the middle of the spark-gap. The receiver was used with its plane (1) vertical, (2) horizontal, in both cases with its centre in the horizontal plane through the vibrator.

It was observed in the former case that no sparks passed when the circle was placed with its diameter through the spark-gap horizontal, that is the gap vertical, but that sparks passed with increasing intensity as the receiver was turned round in its own plane so as to bring this diameter nearer to the vertical. When the gap was at the lowest or highest point of the circle, and therefore horizontal, the sparks passed most freely for a given position of the plane of the circle.

For any vertical position of the circle clearly $\alpha = 0$. For a vertical position of the gap the action of β is equal and opposite in the two halves, for a horizontal position its action on the part of the circle diametrically opposite is effective unless $\psi = 90°$.

It was found that if, when the spark-gap was at the top or bottom of the circle, the receiver was turned round a vertical axis there were two positions in which the sparks passed with maximum intensity, and two in which there was absolute or approximate extinction of the sparks. The two positions of maximum were 180° apart, as were also the two positions of zero, which lay midway between the two former positions. For the former position $\psi = 0$, and $\theta = 90°$ for the element opposite the gap, for the zero positions $\psi = 90°$.

A number of these positions were noted and are illustrated in Fig. 190. The longer lines show the positions of the spark-gap when the sparking was a maximum, the short arrow-pointed lines the direction of the electric force. The short lines clearly indicate curves of electric force, the others directions at right

angles to the electric force. Furthermore the lines which are drawn for points near the vibrator suggest the lines of electrostatic force illustrated in Figs. 180—183, and the theory shows that near the vibrator the electrostatic action is most powerful.

Experiments with Resonator Horizontal.

When on the other hand the plane of the receiver was horizontal, no effect was produced when it was on the position I, Fig. 191, so that the magnetic induction through it was zero, and the spark-gap was at b_1 or b'_1 and therefore at right angles to the electric force. When however the circle was turned round in its own plane so that the spark-gap was brought to a_1 or a'_1, equal maxima of spark production were found at these points. The spark-length observed at these points was 2·5 mm.

When the circle was turned to II the magnetic induction through it was no longer zero. Two positions of minimum or zero sparking were found at b_2 and b'_2, and two maxima of unequal intensity at a_2, and a'_2. The line $a_2a'_2$ was at right angles

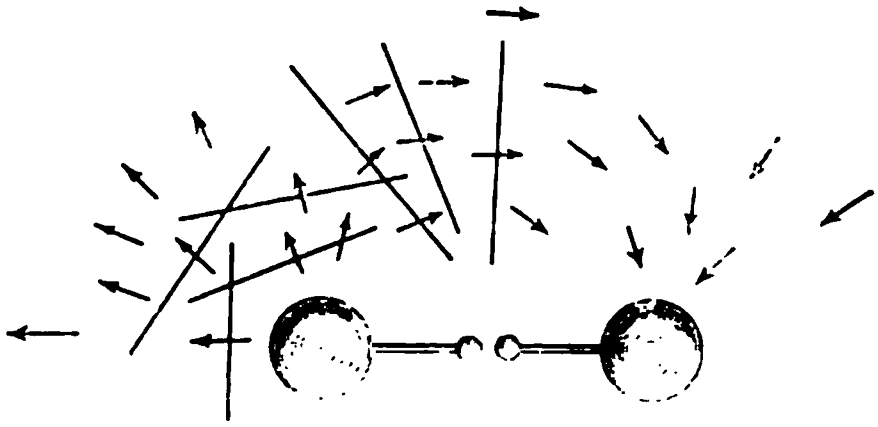


FIG. 190.

to the electric force, and thus the action was proportional to $a + \beta$ at one position, and to $a - \beta$ at the other. The two effects conspired at a_2, and were opposed at a'_2. For the electric force, with A say positively charged, was from A and tended therefore, when the gap was at a_2, to produce a current in the clockwise direction. But the electric force due to inductive action was in the direction from A' to A near the vibrator, and tended to produce flow in the same direction.

When the spark-gap was at a'_2 it is easy to see in like manner that the effects were opposed.

When the spark-gap was at b_2 or b'_2 the electric force being equally inclined to the circle at those points gave therefore equal components along the circle, which neutralized the electric forces due to induction.

The spark-lengths at a_2 and a'_2 were 3·5 mms. and 2·5 mms. respectively.

When the circle was moved to position III the two null points closed up nearer to a'_3 the smaller maximum, while the greater maximum was at a_3. Over a considerable region opposite to a_3 only a very slight effect was observed. The spark-length at a_3 was 4 mms.

As the middle position was approached and reached in IV and V, no positions of extinction at all were found, but only a maximum and minimum at a_4, a'_4, in IV and a_5, a'_5, in V. It is to be noticed that in the passage from position III to position V the line aa' turned quickly round through nearly 90° so as to be always at right angles to the electric force.

The spark-lengths found were 5·5 mms. at a_4, 1·5 mms. at a'_4 and 6 mms. at a_5, 2·5 mms. at a'_5.

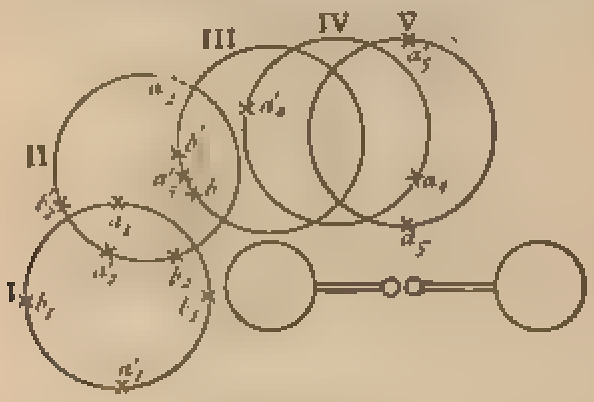


FIG. 191

Resonator inclined to Horizontal.

Hertz made experiments also with the resonator in other positions than those specified, and found the results to be in accordance with theory. For example the circle was placed in the position V of Fig. 191 with the spark-gap at a_5, and then turned round the diameter parallel to the vibrator so as to raise the spark-gap. Experiments were made for various positions until the circle had been turned completely round to its original position. During the change of position θ was still 90°, β remained nearly constant, but a changed with the cosine of the inclination of the plane of the circle to the horizontal. Thus if the value of a for the horizontal position be denoted by a_0, and the inclination of the circle to the horizontal by ϕ, the quantity roughly measuring the spark producing action was $a_0 \cos\phi + \beta$, and varied therefore from $a_0 + \beta$ for the spark-gap at a_5 in the position V, to β when ϕ was 90°. Experiment showed that the spark-length varied from 6 mm. to 2 mm.

Effect of Change of Angle of Inclination.

Thereafter as the next quadrant was turned through the spark-length passed through zero and increased again to the smaller maximum 2·5 mms. at a'_3 when ϕ was 180°. The action was then represented by $-a_0+\beta$, and $-a_0$ preponderating gave the smaller (negative) maximum, while between $\phi = 90°$, and $\phi = 180°$, $a_0 \cos\phi + \beta$ took the value 0.

As ϕ was further changed from 180° to 270° $a_0 \cos\phi + \beta$ changed from $-a_0+\beta$ to β again, and so the spark-length changed through zero to 2 mms. once more. As the circle was turned through the last quadrant to its original position the spark-length increased from 2 mm. to 6 mm.

Experiments at Greater Distances

The experiments just described were all made in the vicinity of the vibrator. Experimenting at greater distances Hertz found that at points from about 1 to 1·5 metre from the vibrator the maximum and minimum positions were not clearly defined

FIG. 192.

except for certain positions; but that they became again distinct at distances exceeding two metres. Fig. 192 shows the electric field as mapped out by Hertz, in an exploration made by means of a receiver carried about from point to point in a room of 14 metres by 12. From this exploration he drew the following conclusions: (1) That at distances beyond three metres the electric force is parallel to the oscillation, and is due in the main to induction. (2) For distances less than 1 metre from the vibrator the electric force is almost wholly electrostatic. (3) The electric force is determinate at all points along the axis of the vibrator and in the equatorial plane but within a certain region, marked by the asterisks in the diagram, becomes in-

Exploration of Electric Field with Resonator.

determinate. The effect falls off much more rapidly with increase of distance along the axis than in the equatorial plane.

It is to be remarked that these results, the explanation of which was not at all obvious when they were observed, are on the whole satisfactorily accounted for by the theory given later by Hertz and discussed above, p. 775 *et seq.*

Results of Exploration referred to Theory.

It is only necessary to look at Figs. 182, 183 to find all the features of Fig. 192. There is the so-called electrostatic field of force close to the vibrator, the region of indeterminacy beyond it, and, in the neighbourhood of the equator at least, the parallelism of the electric force to the vibrator. The more rapid falling off of the action along the axis is also explained. For by equations (13), at a great distance the electric and magnetic forces are both zero on the axis, while in the equatorial plane they are still sensible. The only discrepance, if discrepance it is, is the apparent parallelism of the lines of electric force to the vibrator found at some distance from the equatorial plane; but this is no doubt due to the inaccuracy produced by effects of the walls of the room, or otherwise.

Effect of Conductors and Insulators in Field of Vibrator. Arrangement of Apparatus.

Hertz also investigated the effect of placing conductors and insulators of different kinds in the neighbourhood of his vibrator.* The arrangement of apparatus is shown in Fig. 193. AA' is the exciter consisting as shown of two square plates of brass, 40 cms. in side, placed symmetrically in the horizontal plane about the line mn, and joined by a wire 70 cms. long, interrupted in the centre by a spark gap of $\frac{3}{4}$ cm between two well polished brass knobs. The two sides of the gap were connected as indicated by the wires to the terminals of an induction coil.

The receiver was a circle of the dimensions already specified and furnished with a screw at the gap which enabled the spark-length to be varied from a few hundredths of a millimetre to several millimetres. The circle was set with its plane vertical and parallel to the spark-gap, and its centre on the horizontal line mn, the axis of symmetry of the vibrator, at a distance of 12 cms. from the nearest points of the plates. It was made movable round an axis coinciding with this line, so that the position of the spark gap could be varied on the circle.

When the spark-gap f in the receiver was at a or a' no sparks passed, but the slightest turning from that position caused the sparks to begin, and they had a maximum length of about

* *Wied. Ann.* 34 (1888) p. 273, or *Untersuch. ü. d. Ausbreitung*, &c. p. 102.

3 mms. when the spark-gap was at b or b'. For this position the sparks were of course produced by the electric force independently of any change of the total magnetic induction through the circle, since of course that was zero. It was found that if without changing the plane of the circle it was placed at a lower level the spark at b increased, and that at b' diminished in length, and the places of zero spark were no longer at the extremity of the diameter of the circle at right angles to bb', but were both displaced on the circle towards b to an equal amount. This result was due to magnetic induction, the integral of which through the circle was no longer zero. There is no difficulty in working out the explanation in detail.

Precautions in Using Apparatus. It was found necessary to set up the apparatus in a large chamber at a great distance from all other objects, and to keep conducting bodies such as bars of metal at a distance from the apparatus, in order that the absence of sparks at a and a' might be maintained. An unsymmetrical position of the body of the observer relatively to mn was quite sufficient to affect the production of sparks.

Effect of Conductors. Effect of Forced Vibrations produced in a Secondary Exciter. A conductor of the shape indicated by C, Fig. 193 was placed above the exciter as shown. When it was brought nearer to the exciter the sparks diminished at b' in the resonator, and increased at b, and the points of zero sparking moved upwards towards b', while sensible sparks appeared at the former zero points. It is easy to see that this effect was what was to be expected. For the conductor C having been made so as to have a natural period shorter than that of the vibrator was the seat of a vibration opposed to that in the primary. This was a consequence of the inductive action on the primary, for since the natural period of the primary was greater than that of C, the electrification of C agreed in phase with the inductive action producing it (see p. 798 above). That this was the explanation Hertz satisfied himself by replacing the vertical connecting plate in C by a thin wire, that by a thinner and so on. The result was to bring the points of zero sparking nearer to the top. But as they moved upwards with the substitution of thinner and thinner wires the zero points disappeared altogether and were replaced by places of minimum sparking. The spark-length up to the vanishing of the zero points was much smaller at b than at b', but at the vanishing of the zero points it began again to increase, and as the period of C was increased the sparks became equal, but no point of zero sparking was again found. As the period of C was increased beyond this stage the spark-length at the lowest point diminished, and there were developed in its neighbourhood two minimum points which

became more and more nearly zero points, and approached at the same time the positions a a'.

Discussion of Results.

All this was of course in strict accordance with theory, for as the period of C approached that of AA' the current in C increased likewise, but when the period was carried beyond that of AA' a difference of half a period was set up between the current in C and the action producing it, that is the two currents agreed in phase. When however the stage of resonance was nearly

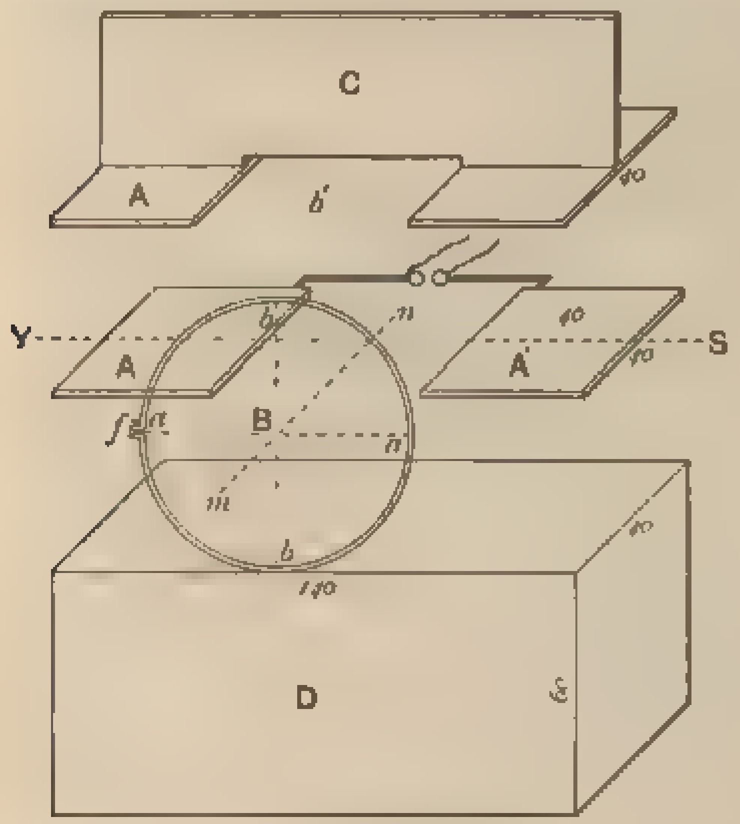


Fig. 193.

reached or only slightly passed the difference of phase was, as in all cases of slightly damped vibration, about a quarter of a period (see (38) p. 798) and interference between the effects of C and AA' was then impossible. This was the stage at which the sparks were equal at the highest and lowest points of the circle.

When C was brought quite close to AA' the sparks in the resonator became small. This was due to the fact that the

capacity of AA' being increased, its period was lengthened, and so the resonator and exciter were thrown out of unison.

Effect of Placing Insulators in the Field.

The effect of placing insulators of different kinds under the exciter was then tried. A pile of books 1·5 metres long, ·5 metre broad, and 1 metre high was placed under the exciter so that its upper surface almost touched the plate. Sparks now passed when the spark gap of the resonator was in its former position of no sparking, and the spark gap had to be turned round the circle through 10° towards the pile of books to reduce the sparking to zero.

Effect of Block of Asphalt placed near Exciter.

A block of asphalt of the dimensions (in cms.) indicated in Fig. 194 was then placed in the position shown. The sparks were now found to be much stronger at b' than at b' the point nearest the asphalt. The points of zero sparking ceased to exist, and minimum points were then found below the level of the former zero points, an angular distance on the circle of about 23°.

When the plates AA' were laid on the asphalt the period of the vibrator was increased, as was shown by its being necessary to increase the period of the receiver to cause it to respond.

By increasing the distance between the apparatus and the block of asphalt the effect of the latter was diminished, but its character remained unchanged.

It was found possible to compensate the action of the asphalt by bringing down the conductor C sufficiently near to the vibrator AA'.

The experiments were repeated with a large number of other substances, such as wood, sandstone, sulphur, paraffin and petroleum.

Phenomena not due to Electrostatic Action.

According to Hertz these results prove that change of electric force in a dielectric is always accompanied by a corresponding magnetic action. It is impossible to account for them by electrostatic action pure and simple. For if the exciter be placed with its line rs at the middle of and along one edge of a large rectangular block of insulating material, there can be no change of electrostatic action in the space, for example, in front of the block and below the plane of the vibrator. Still the effect was found by Hertz to be quite marked in this space.

Again the effects cannot be due to conduction currents in the dielectric, for both sulphur and paraffin showed the effects in a very marked degree.

There remains therefore only the change of magnetic field and the consequent inductive electric force at each point to explain the phenomena.

It is a result of electromagnetic theory that if the variation of electric action in an insulating medium is accompanied by

magnetic action according to the laws of induction laid down above electric action must be transmitted in the medium with a finite velocity. That the two go hand in hand is the foundation of Maxwell's theory of electrodynamics, and Hertz's experiments just described gave evidence of the truth of the theory for solid dielectrics. It remained still to show that it was true for air. Hertz attempted to verify the finite velocity in air by carrying the secondary vibrator C from a position of interference with AA' to a greater and greater distance and observing whether the interference ceased and began again alternately. This experiment however was not successful, but the problem was solved by Hertz in another manner

Determination of Velocity of Propagation in Air.

It was known that electric oscillations could be propagated along a wire, and attempts made by Fizeau and Gounelle in 1850, and by W. Siemens in 1876 to determine the velocity of propagation for iron and copper wires gave velocities of the same order of magnitude as the velocity of light.

Interference between Waves in Air and Waves along Wires proves Finite Velocity in Air.

Arrangements were therefore made to compare the finite velocity of electric oscillations in air, if such a velocity existed with the velocity of propagation in a wire.* An exciter consisting as before of two 40 cms. square brass plates joined by a 70 cm. long brass rod with a spark-gap between was set up as shown in Fig. 194. Behind the plate A was arranged a plate P connected as shown to a long wire of copper 1 mm. thick carried along horizontally 30 cms. above the horizontal axis of symmetry of the vibrator. This axis we shall call the base line of the apparatus. A point 45 cms. from the spark-gap of the exciter was taken as zero point for the measurement of distances parallel to the wire.

Arrangement of Apparatus.

The resonator used was either the circle of 35 cms radius or the square of 60 cms. side already described. It was placed in three different positions with its centre on the base line:—(1) in the vertical plane through the base line; (2) with its plane at right angles to the base line; (3) with its plane horizontal.

By the direct action of the exciter no sparks were produced in the first position, while in the second position sparks only occurred when the gap was above or below the level of the base line. The cause of this sparking was, as we have seen, the electric force acting on the part of the receiver opposite the air-space. The total magnetic induction through the circle in both

Experimental Results. Direct Effect of Exciter.

* *Sitzungsbr. d. Berl. Akad. d. Wiss.* 2 Febr. 1888; *Wied. Ann.* 34 (1888) p. 551, or *Untersuch. u. d. Ausbr.* &c.

positions was zero. The length of the sparks fell off with distance from the exciter, at first rapidly, then much more slowly, and they were visible up to 12 metres from the exciter.

In the third position sparking was obtained no matter how the spark-gap was placed. The spark-length was a maximum when the gap was parallel to the exciter and on the side nearest to the latter, and a minimum when the gap was in the diametrically opposite position. These results were of course due to the joint action of the electric force and the magnetic induction. The effect of the latter was of course the more powerful as there was no zero of sparking.

Effect of Waves in Wire on Resonator.

Considering now the action of the wire, we see that the arrangement adopted insured that the period of the disturbance should be the same as that produced directly by the exciter. When the action of the wire alone on the resonator was observed reflected waves were avoided by continuing the wire to a distance of 60 metres, and terminating it in an earth connection. Sparks were found to pass with greatest freedom between the knobs when the receiver was placed in a plane through the wire, and had its spark-gap as close to the wire as possible.

Phenomena of Standing Oscillation.

A standing oscillation was produced in the wire by arranging it with a free end at which reflection could take place, and it was then found by moving the receiver along the wire that places of maximum and zero sparking existed alternately at equal intervals along the wire, that is to say nodes and loops of standing electrical waves were indicated. The nodes were places of maximum or minimum potential but of no flow of electricity, consequently electric forces are directed outwards from the nodes.

Hence it was seen that if the receiver were placed with its plane at right angles to the wire and its spark-gap in an intermediate position between that nearest to the wire and that farthest from it, these electric forces ought to produce sparking. This was found to be the case. Again sparks were readily obtained from a node by bringing near it any small conductor. These effects were however slight as the disturbance in the wire was really a complex of vibrations of different periods from which the resonator picked out those of its own proper period as in the analogous acoustic experiment.

Effects independent of Thickness of Wire.

Experiments on wires of different thicknesses showed that no change was produced in the positions of the node and loops. Even in an iron wire these remained unaltered, showing that with the rapid changes here in question practically no magnetization was produced.

The nodes and loops were most clearly produced when the

wire was cut 8 metres or $5\frac{1}{2}$ metres from the zero point of the base line. In the former case the nodes were at ·2 metre, 2·3 metres, 5·1 metres, 8 metres from the zero point; in the latter ·1 metre, 2·8 metres, 5·5 metres from the same point. These results indicate clearly a half wave-length of 2·8 metres. The period of oscillation as calculated from an estimate of the inductance and capacity of the exciter was about 2×10^{-10} cms. per second. This gave a velocity of propagation of $2\cdot 8\times 10^{10}$ cms. per second, that is approximately the velocity of light.

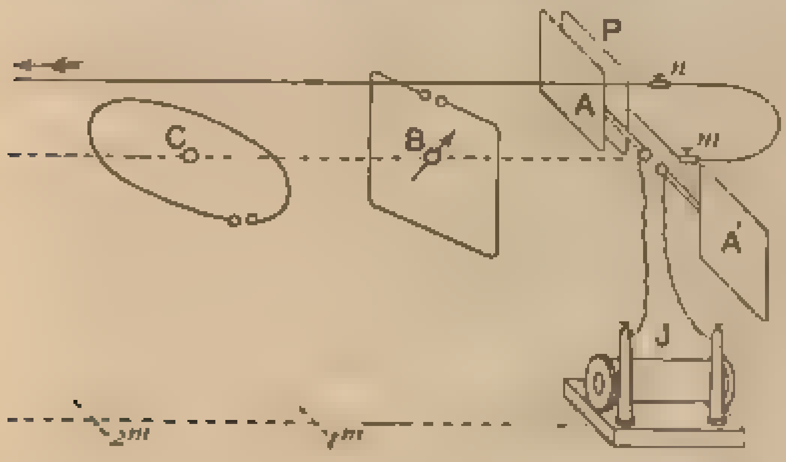


FIG. 194.

Arrangement of Resonator for Interference. The receiver could be placed so that both the exciter and the wire could produce sparks in it. For this it was necessary that it should be placed neither in the first position nor in the second, but in an intermediate position. Thus the square receiver was placed at the zero point with its plane vertical, and the spark-gap at the top, and the normal to its plane turned towards the plate A or the plate A' of the exciter. Let us suppose that it pointed towards A as shown in Fig. 194, and that A had its maximum positive charge. Then the direct electric force due to the exciter and the inductive electromotive force at the node in the wire were opposed to one another. When the receiver was turned round so that its plane pointed towards A' the sign of the direct action on the receiver was changed and the two effects conspired. By moving the plate P to a greater or less distance from A the effects of the wire could be varied in strength so as to be made equal to the direct action on the receiver.

Interference effects could also be obtained with the receiver in the third position, but in this case the wire had to be on one

[redacted]

side or the other of the base-line. Experiment showed that, when the wire was on the same side of the base-line as P, the effect of the waves in the wire was opposed to that of the exciter, and when the wire was transferred to the opposite side the effects conspired. The position of the spark-gap did not affect the nature of these results, which showed that it was mainly inductive action in both cases which produced sparks.

Observations of Interference. Using the first mode of observing interference Hertz included different lengths of wire as the arc between the points mn, and found that as the length was increased the origin changed from a node to a loop, then back again to a node and so on. It was found however that if at one node the effects were opposed when the normal to the receiver pointed to P, at the next node they were opposed with the normal turned away from P and so on alternately.

The same alternation of the direction of the normal was found at the different nodes along the wire for a given length of wire between mn.

The following are some of the observations made. A distance of 8 metres from the zero point was laid off along the wire, and the receiver set up at every ½ metre of this distance and its normal placed successively in the two positions specified and the difference if any in the spark-length noted. If no difference was observed the result was indicated by the sign O, or according as the sparks were smaller or greater with the normal turned towards P than in the other case the result was marked by + or −. Eleven series of such observations were made along the 8 metres of wire with an additional 50 cms of wire between mn for each series, so that the length of wire increased from 100 to 600 cms.

Results: Wave-length of Air. The results showed that the sign of the interference changed for successive displacements of between 3 and 4 metres along the wire, and also for about the same length of wire introduced between m and n. This showed that the wave-length of the direct effect was not the same as that in the wire, and that the velocity of propagation in air though finite was greater than that in the wire.

A second set of observations was made along a length of 12 metres of the wire with, as before, different lengths of wire included between m and n. These results being complicated with electrostatic action near the exciter, a series of observations were made up to 4 metres with the receiver in the third position and the wire on one side as already explained. The results of the latter experiments taken along with those for the remaining 8 metres given by the other method gave the following table of results.

Length of Wire between m and n in metres.	Distance along Wire from Origin in Metres.												
	0	1	2	3	4	5	6	7	8	9	10	11	12
1	+	0	−	−	0	0	0	+	+	+	+	+	0
2·5	0	−	−	0	+	+	0	0	0	0	−	−	−
4	−	0	+	+	0	0	−	−	−	−	0	0	0

This corroborates the conclusion from the former series of results.

Velocity of Propagation in Air.

It appears therefore that the wave in air while traversing 7·5 metres gained half the length of the wave in the wire on the latter: that is while the wave in air traversed 7·5 metres the wave in the wire traversed 7·5 − 2·8 (= 4·7) metres. But the half-period of the wave in the wire being 1×10^{-8} second the velocity of propagation of the wave in air given by the experiments was $4 \cdot 5 \times 10^{10}$ cms. per second. This exceeds the velocity of light by 50 per cent. of the latter and gives a wave length in air of about 9 metres.

This result, it may be stated here, does not agree with the observations of later experimenters who have found for waves in air as well as for waves in wires the velocity of light. The discrepancy will be discussed later.

Reflection of Waves

Hertz also made experiments on the reflection of waves in air from conducting surfaces.* The experiments were carried out in his physical lecture theatre, a room about 15 metres long, 14 metres wide, and 6 metres high. Parallel to the side walls were two rows of iron columns so that the clear breadth of the room was about 8·5 metres. All gasaliers and other removable obstacles were cleared away, and one end wall from which the reflection was to take place was covered with a plate of zinc 4 metres high and 2 metres broad connected by wires to the gas and water pipes. The exciter was set up two metres from the opposite end of the room with its axis vertical. The waves were incident nearly normally on the plate of zinc, and the electrical vibration was therefore in the vertical plane through the vibrator.

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The receiver, the circle of 35 cms. radius already described, was carried along the normal through the centre of the vibrator, and the positions of maximum and minimum sparking in the neighbourhood of the wall observed. The positions I, II, III, IV, in the diagram were those of strongest sparking. In these, it will be seen, the spark-gap was turned alternately in opposite directions. The arrows show by their directions and lengths the electric forces on the two sides of the circle, and explain the result.

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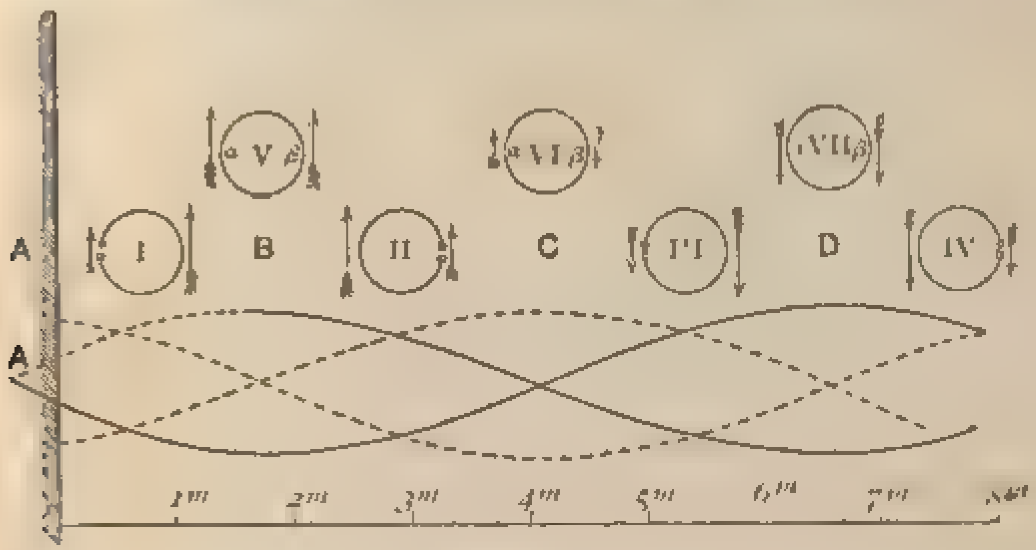
All the results are explained if we suppose that standing waves of electric and magnetic force are produced, as represented by the full and dotted curves in Fig. 195. It is shown by the theory given above (see equations (18) p. 786), that at a distance from the vibrator the electric and the magnetic forces are propagated together in the same phase. The diagram therefore shows that in the act of reflection the electric force has its phase changed by half a period relatively to the magnetic force, so that in the standing vibrations the nodes of one correspond to the loops in the other, and *vice versa*. Explanation of Results.



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The observations seemed to indicate that the node for the electric force was behind the wall surface about ·68 metre, and the next loop but one, about 6·52 metres in front of it, so that the wave-length was about 9·6 metres. With the period 2×10^{-8} second for the vibrator this would give $4{\cdot}8 \times 10^{10}$ cms. per second as the velocity of propagation of the waves in air. Velocity in Air given by Experiments Exceeds Velocity of Light

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Possible Explanations of Discrepance. Multiple Resonance.

It has been suggested that the wave-length observed may depend to a great extent on the dimensions of the resonator, and may be connected with what has been called *multiple resonance* by Messrs. Sarasin and De la Rive. It has been noticed by these experimenters, as well as by Fitzgerald and Trouton, that the exciter apparently gives rise neither to a single vibration of distinct period nor to a limited number of distinct vibrations, but rather to such a complex of vibrations as would give a wide band of continuous spectrum. Thus all vibrations, agreeing with possible modes of vibration of the resonator, would be reinforced. That this is not contained in theory is true, but the theory is very incomplete. It is hard to believe that the vibrations can be perfectly simple.

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It may be noted here that it has been found by Mr. Trouton * that the size of the reflecting sheet has a great deal to do with the distance of the nodes from the surface. Using long narrow strips held (1) so that the length was in the direction of the magnetic component; (2) in the direction at right angles to that component, he found that the node was in the former case shifted outwards from the reflecting surface very markedly. For example, with waves 68 cms. long the distance of the magnetic node varied from 24·2 cms. for a strip 16 cms. wide to 17 cms. ($\frac{1}{4}$ wave-length) for a large sheet. This effect was due no doubt, as stated by Mr. Trouton, to the action of the charge periodically accumulated at the edges of the sheet.

Smallness of size in the magnetic direction carried the node in towards the surface; and this may very possibly have been the case in the experiments of Hertz described above. The breadth of the sheet (in the direction of the magnetic force) was 2 metres, or about the same in effect as a strip 14 cms. broad

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Experiments were also made by Hertz on the production of plane polarized waves, by means of a linear vibrator consisting of two cylinders placed in line with a spark-gap between their opposed ends. The cylinders were about 12 cms. long and 3 cms. in diameter each, and the ends at the spark-gap were well rounded. The vibrator was placed vertically in the focal line of a parabolic cylindrical reflector made of ordinary sheet zinc nailed on a wooden framework cut into proper parabolic

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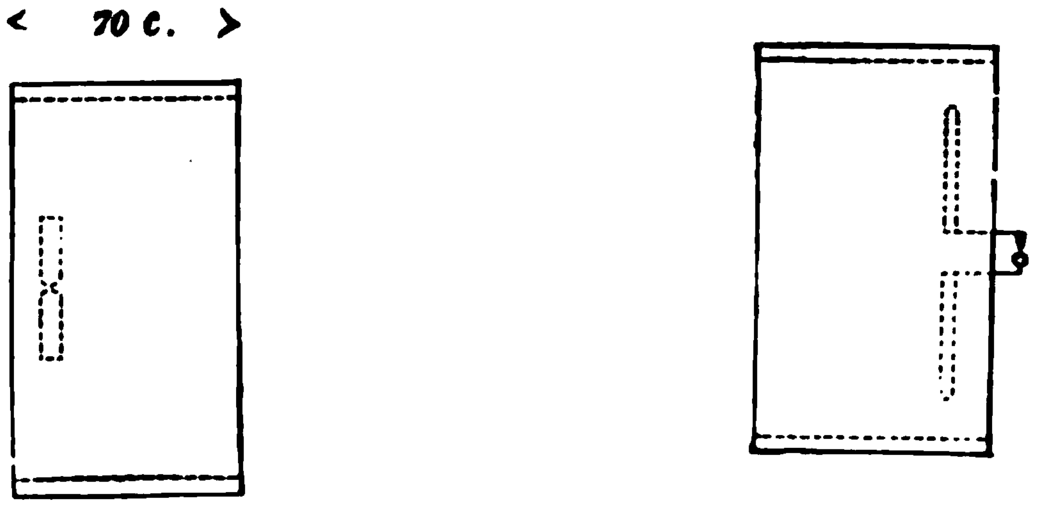

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The Receiver. In the focal line of the other mirror was placed a receiver made of two pieces of thick wire each 50 cms. long placed in line as shown in Fig. 197, with a gap of about 5 cms. between their ends, and completed by two thin wires about 12 cms. long led out at right angles to the rods to the back of the mirror. These were tipped with a knob and point as shown, so as to form an adjustable spark-gap which could be conveniently observed from behind.

It was found by this arrangement that electric radiation could be detected at a much greater distance from the source than with the ordinary vibrator and receiver used as described above without reflectors. In these as in all other experiments the knobs of the vibrator have to be repeatedly cleaned and its spark-gap must be screened from the direct light of the spark in the induction coil.

Clearly a parallel beam of plane polarized light was thus

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Hertz found that such an electromagnetic wave was not only reflected like a light wave, but is also refracted according to the same law of refraction. An immense prism of pitch having an isosceles triangular section of 120 cms. side, and a refracting angle of 30°, was made by melting pitch into a wooden supporting case. The prism was placed, with its refracting edge vertical, at a distance from the vibrator of 2·6 metres, and the beam was made incident on the face at an angle of 65°. The receiving mirror was estimated 2·5 metres from the prism on the other side, and showed a radiation beginning, reaching a maximum, and falling off to zero, at deviation 11°, 22°, 34°.

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The experiments were repeated with the focal lines of the mirror parallel, and practically no difference in the result was observed.

The index of refraction for pitch given by the experiments was 1·69, which nearly agrees with the index 1·5 to 1·6 found for pitchy substances by optical experiments.

Prof. Oliver Lodge and Dr. Howard have made observations on the concentration of such vibrations by means of lenses.† Two enormous lenses of hyperbolic cylindrical figure were con-

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structed of mineral pitch, and were placed with their axial planes coincident, and their plane faces, or bases, turned towards one another. These lenses were so proportioned that the beam produced by a linear exciter in the external focal line of one might emerge parallel from the plane face of that lens, and then be concentrated by the second lens on the corresponding focal line.

Optically Opaque Bodies Transparent to Electrical Waves

An interesting point noticed in many of these experiments is the perfect transparency to these vibrations of optically opaque substances. A stone wall three feet thick has been found to offer no obstacle to the passage of such waves. In fact in some experiments made by Prof. Fitzgerald at Dublin the receiver was placed on a pillar in the garden outside, while the exciter was in action in the laboratory. This is no doubt a phenomenon of the same character practically as that of the transparency of a thin film of metal to light, and is conditioned by the relation of the wave-length to the thickness of the obstacle. [See p. 200 above. It has been found by Maxwell, *El. and Mag.* Vol II. Chap. XIX., that the transparency of thin metallic films is greater than that given by the electromagnetic theory. See also Wien, *Wied. Ann.*, 35 (1888).]

Much interesting information regarding electrical radiation has been obtained by Trouton, Boys, Dragoumis, and others, of which it is impossible here to give any account. The great desideratum however now is some method of maintaining electrical vibrations, so as to enable their phenomena to be fully studied.

Lodge's Experiments on Electrical Discharge and Radiation.

In concluding this chapter and the present work reference must be made to the great mass of extremely interesting and important results connected with electrical discharge and radiation which have been obtained during the last four or five years by Dr. Lodge of Liverpool.* He has shown that in many cases a thunder cloud discharging to the earth is really a great Hertzian vibrator, and that the discharge, being therefore oscillatory in character, produces most violent electrical surgings in metallic conductors, whether insulated or otherwise, which happen to be within a moderate distance of the discharge.

Construction of Lightning Conductors.

These results of theory and experiment have led to important conclusions as to the proper construction of lightning conductors for such cases. Thus Dr. Lodge has pointed out the

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importance of surface in a lightning conductor since, in a rapidly varying discharge the current is practically confined to the surface stratum, and the superiority from this point of view of iron over cotton.

In consequence of the "circular" magnetization of an iron wire produced by a current along it, the current is more strictly confined to the surface than in a copper conductor, so that the increase of resistance thus produced is sufficient to insure a more rapid dissipation in an iron wire of comparatively small cross-section. Whether it would be safe to make all lightning conductors according to this principle is a question about which there is considerable difference of opinion; but there can be no doubt that it will not do any longer to consider a lightning discharge as a mere case of ordinary conduction to be provided for simply by a large thickness of good conducting material.

It has been found, also, by V. Bjerknes (*Electrician*, Nov. 18, 1892) that the damping out of oscillations, excited in a Hertzian resonator, takes place more quickly when the resonator is made of iron, than when it is made of non-magnetic material. This agrees with the result just stated.

side or the other of the base-line. Experiment showed that, when the wire was on the same side of the base-line as P, the effect of the waves in the wire was opposed to that of the exciter, and when the wire was transferred to the opposite side the effects conspired. The position of the spark-gap did not affect the nature of these results, which showed that it was mainly inductive action in both cases which produced sparks.

Observations of Interference. Using the first mode of observing interference Hertz included different lengths of wire as the arc between the points mn, and found that as the length was increased the origin changed from a node to a loop, then back again to a node and so on. It was found however that if at one node the effects were opposed when the normal to the receiver pointed to P, at the next node they were opposed with the normal turned away from P and so on alternately.

The same alternation of the direction of the normal was found at the different nodes along the wire for a given length of wire between mn.

The following are some of the observations made. A distance of 8 metres from the zero point was laid off along the wire, and the receiver set up at every ½ metre of this distance and its normal placed successively in the two positions specified and the difference if any in the spark-length noted. If no difference was observed the result was indicated by the sign O, or according as the sparks were smaller or greater with the normal turned towards P than in the other case the result was marked by + or −. Eleven series of such observations were made along the 8 metres of wire with an additional 50 cms of wire between mn for each series, so that the length of wire increased from 100 to 600 cms. The results showed that the sign of the interference changed for successive displacements of between 3 and 4 metres along the wire, and also for about the same length of wire introduced between m and n. This showed that the wave-length of the direct effect was not the same as that in the wire, and that the velocity of propagation in air though finite was greater than that in the wire.

Results: Wave-length of Air.

A second set of observations was made along a length of 12 metres of the wire with, as before, different lengths of wire included between m and n. These results being complicated with electrostatic action near the exciter, a series of observations were made up to 4 metres with the receiver in the third position and the wire on one side as already explained. The results of the latter experiments taken along with those for the remaining 8 metres given by the other method gave the following table of results.

Length of Wire between m and n in metres.	Distance along Wire from Origin in Metres.												
	0	1	2	3	4	5	6	7	8	9	10	11	12
1	+	0	−	−	0	0	0	+	+	+	+	+	0
2.5	0	−	−	0	+	+	0	0	0	0	−	−	−
4	−	0	+	+	0	0	−	−	−	−	0	0	0

This corroborates the conclusion from the former series of results.

Velocity of Propagation in Air.

It appears therefore that the wave in air while traversing 7·5 metres gained half the length of the wave in the wire on the latter: that is while the wave in air traversed 7·5 metres the wave in the wire traversed 7·5 − 2·8 (= 4·7) metres. But the half-period of the wave in the wire being 1×10^{-8} second the velocity of propagation of the wave in air given by the experiments was $4{\cdot}5 \times 10^{10}$ cms. per second. This exceeds the velocity of light by 50 per cent. of the latter and gives a wave length in air of about 9 metres.

This result, it may be stated here, does not agree with the observations of later experimenters who have found for waves in air as well as for waves in wires the velocity of light. The discrepancy will be discussed later.

Reflection of Waves

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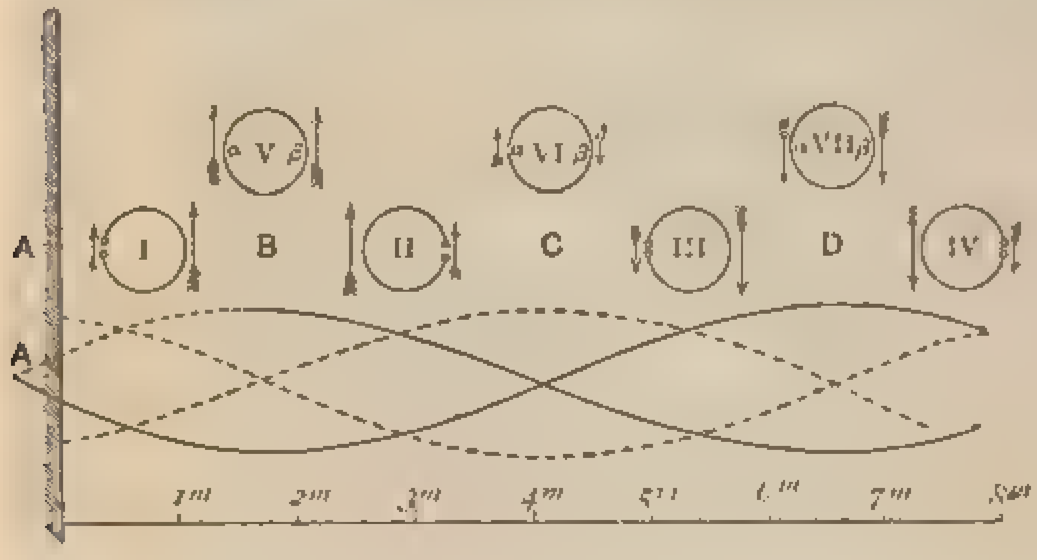

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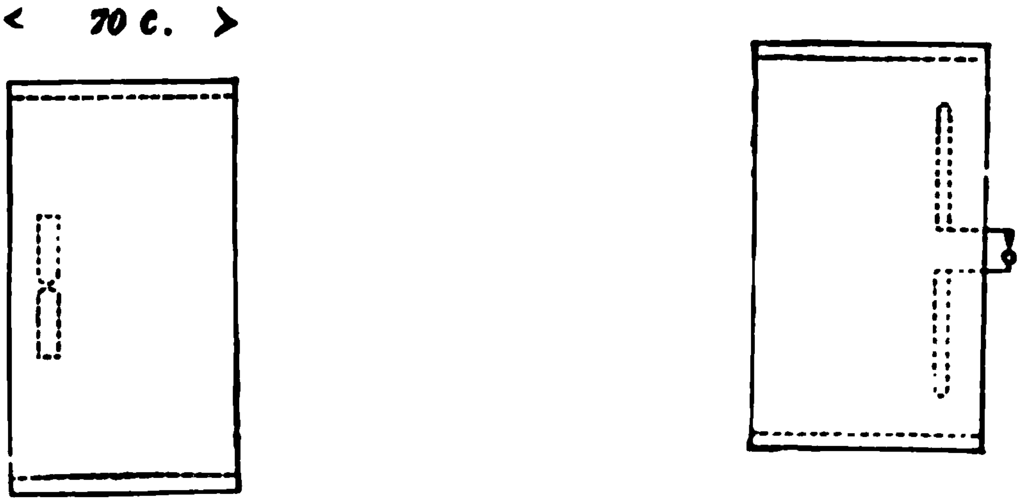

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* See a series of papers in the *Electrician*, vols. 22, 23 (1888–9, 1889), and a lecture on "Electrical Oscillations," Royal Institution, April, 1889, *Electrician*, vol. 22.

importance of surface in a lightning conductor since, in a rapidly varying discharge the current is practically confined to the surface stratum, and the superiority from this point of view of iron over cotton.

In consequence of the "circular" magnetization of an iron wire produced by a current along it, the current is more strictly confined to the surface than in a copper conductor, so that the increase of resistance thus produced is sufficient to insure a more rapid dissipation in an iron wire of comparatively small cross-section. Whether it would be safe to make all lightning conductors according to this principle is a question about which there is considerable difference of opinion; but there can be no doubt that it will not do any longer to consider a lightning discharge as a mere case of ordinary conduction to be provided for simply by a large thickness of good conducting material.

It has been found, also, by V. Bjerknes (*Electrician*, Nov. 18, 1892) that the damping out of oscillations, excited in a Hertzian resonator, takes place more quickly when the resonator is made of iron, than when it is made of non-magnetic material. This agrees with the result just stated.

APPENDIX

I.—ZONAL SPHERICAL HARMONICS.

Spherical Harmonic Defined.

A spherical harmonic may be defined as a homogeneous function of x, y, z which satisfies Laplace's equation.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since it is homogeneous it satisfies also the relation

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = nV \quad . \quad . \quad . \quad . \quad . \quad (2)$$

if n be the degree of the function.

The fundamental equation may be transformed by the substitution of the variables r, θ, ϕ, connected with x, y, z by the equations

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Of these θ may be regarded as the co-latitude and ϕ the longitude, or θ and ϕ may be taken as respectively the polar distance and right ascension of the point x, y, z, of which r is in both cases the radius vector from the origin.

Laplace's Equation in Polar Co-ordinates.

When these substitutions are made Laplace's equation becomes

$$r\frac{\partial^2(rV)}{\partial r^2} + \frac{1}{1-\mu^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial V}{\partial \mu}\right\} = 0 \quad . \quad (4)$$

if μ denote $\cos\theta$.

Equation (2) becomes plainly

$$\frac{\partial V}{\partial r} = \frac{n}{r} V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The last result gives

$$r \frac{\partial^2 (rV)}{\partial r^2} = n(n+1)V.$$

Hence (4) takes the form

$$\frac{1}{1-\mu^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial V}{\partial \mu}\right\} + n(n+1)V = 0 \quad . \quad (6)$$

Spherical Surface Harmonic Defined.

If V denote a spherical harmonic of degree n, we may write it in the form $r^n S_n$. S_n is a function of θ, ϕ, but not of r, and is called a spherical surface harmonic of degree n. It satisfies by (6) the equation

$$\frac{1}{1-\mu^2}\frac{\partial^2 S}{\partial \phi^2} + \frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial S}{\partial \mu}\right\} + n(n+1)S = 0 \quad . \quad (7)$$

If $r^n S_n$ denote a spherical harmonic of degree n, $r^{-(n+1)}S_n$ denotes a spherical harmonic of degree $-(n+1)$. To prove this we have only to notice that it clearly satisfies (6), since S_n satisfies (7). Again if we denote it by V, we have

$$\frac{\partial V}{\partial r} = -(n+1)r^{-(n+2)}S_n = -\frac{n+1}{r}V$$

which is what (5) becomes when n is changed to $-(n+1)$.

If S_n is symmetrical about an axis it is called a zonal surface harmonic (or simply a zonal harmonic) of order n. We may take the axis of symmetry as axis of z, so that the symmetry is expressed by making S_n independent of ϕ. We shall denote a zonal harmonic of order n by Z_n.

Differential Equation Satisfied by Zonal Surface Harmonic.

The differential equation satisfied by Z_n is by (7)

$$\frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial u}{\partial \mu}\right\} + n(n+1)u = 0 \quad . \quad . \quad . \quad . \quad (8)$$

The discovery of zonal harmonics resolves itself then into finding particular solutions of this equation. The most import-

ant case, and the only one which we here consider, is that in which n is a positive integer.

Calculation of Zonal Harmonics.

We assume first that u may be expanded in a series of powers of μ. Thus writing

$$u = A_1\mu^{m_1} + A_2\mu^{m_2} + \dots \quad . \quad . \quad . \quad . \quad . \tag{9}$$

substituting in the differential equation (8), and equating coefficients of like powers of μ we get first from those of μ^{m_1}

$$(m_1 - n)(m_1 + n + 1)A_1 = 0.$$

Since A_1 is not zero this gives $m_1 = n$, or $m_1 = -(n+1)$. Thus there are two solutions according as m_1 is taken $= n$, or $= -(n+1)$. m_2 is then found to be $m_1 - 2$, $m_3 = m_1 - 4$, &c.

Again the successive coefficients in (9) are found to be connected by the relation

$$A_r = -\frac{(m_1 - 2r + 4)(m_1 - 2r + 3)}{2(r-1)(2m_1 - 2r + 3)}A_{r-1}$$

whichever value is given to m_1.

Hence if we take $m_1 = n$, (9) becomes

$$u = A_1\left\{\mu^n - \frac{n(n-1)}{2\,.\,(2n-1)}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\,.\,4\,.\,(2n-1)(2n-3)}\mu^{n-4} - \dots\right\} \quad . \quad . \quad (10)$$

Expression for Zonal Surface Harmonic.

The series within brackets in (10) is finite and has for last term $(-1)^{\frac{1}{2}n}\, n!\, n!\, n!/(\frac{1}{2}n!\,\frac{1}{2}n!\,2n!)$ if n be even, and $(-1)^{\frac{1}{2}(n-1)}\mu\, n!\, n!\,(n-1)!/\{\frac{1}{2}(n-1)!\,\frac{1}{2}(n-1)!\,(2n-1)!\}$ if n be odd.

Another series is obtainable by putting $m_1 = -(n+1)$. This and the former multiplied each by an arbitrary constant and added together give the complete solution of (8).*

The series in (10) with $2n!/2^n(n!)^2$ substituted for A_1 is what is called the zonal surface harmonic of order n. Thus

* For a full discussion of the solutions of (8) see Forsyth's *Differential Equations*, §§ 89—99.

$$Z_n = \frac{2n\,!}{2^n n\,!\,n\,!}\left\{\mu^n - \frac{n(n-1)}{2(2n-1)}\mu^{n-2}\right.$$

$$\left. + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}\mu^{n-4} - \ldots\right\} \quad . \quad . \quad (11)$$

Rodrigues' Theorem.

It may be verified by differentiation that

$$Z_n = \frac{1}{2^n n\,!}\frac{d^n}{d\mu^n}\{(\mu^2 - 1)^n\} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and by expansion of $(1 - 2\mu h + h^2)^{-\frac{1}{2}}$ in powers of h that Z_n is the coefficient of h^n in the resulting series. It is this latter fact that renders the choice of the value above assigned to A_1 convenient.

By means of (11) we can at once write down the zonal surface harmonic for any assigned value of n. Thus for values of n from 0 to 7—

$$Z_0 = 1, \qquad Z_1 = \mu, \qquad Z_2 = \tfrac{3}{2}\mu^2 - \tfrac{1}{2},$$

$$Z_3 = \frac{5}{2}\mu^3 - \frac{3.1}{2}\mu, \qquad Z_4 = \frac{7.5}{2.4}\mu^4 - \frac{5.3}{2.2}\mu^2 + \frac{3.1}{2.4},$$

$$Z_5 = \frac{9.7}{2.4}\mu^5 - \frac{7.5}{2.2}\mu^3 + \frac{5.3}{2.4}\mu,$$

$$Z_6 = \frac{11.9.7}{2.4.6}\mu^6 - \frac{9.7.5}{2.4.2}\mu^4 + \frac{7.5.3}{2.4.2}\mu^2 - \frac{5.3.1}{2.4.6},$$

$$Z_7 = \frac{13.11.9}{2.4.6}\mu^7 - \frac{11.9.7}{2.4.2}\mu^5 + \frac{9.7.5}{2.4.2}\mu^3 - \frac{7.5.3}{2.4.6}\mu.$$

A numerical table of the first seven zonal surface harmonics calculated by Prof. Perry for values of μ for every degree from 0 to 90° is given at the close of this *note*.

Maxwell's Method of Axes.

The following method of defining a solid spherical harmonic is due to Clerk Maxwell (*El. and Mag.* Vol. I., Chap. ix.). Let an electric doublet of moment Φ_1 be placed at the origin with its axis in any direction the cosines of which are l, m, n, then by

(8) p. 14 above its potential at the point (x, y, z) at distance r from the origin is

$$V_1 = -\Phi_1\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y} + n\frac{\partial}{\partial z}\right)\frac{1}{r}$$

$$= \Phi_1\left(l\frac{x}{r} + m\frac{y}{r} + n\frac{z}{r}\right)\frac{1}{r^2}.$$

If then the operation $l\partial/\partial x + m\partial/\partial y + n\partial/\partial z$ be denoted by d/dh_1, where h_1 is a distance along the axis, we may call the operation differentiation with respect to the axis h_1 and we have

Doublets of Different Orders and Axes.

$$V_1 = -\Phi_1\frac{d}{dh_1}\left(\frac{1}{r}\right) = \Phi_1\frac{\mu_1}{r^2} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where μ_1 is the angle between the direction of h_1 and of the line drawn from the origin to (x, y, z).

With respect to this kind of differentiation we may notice that if the suffix j indicate any axis whatever with direction cosines l_j, m_j, n_j, and μ_j denote the cosine of the angle between the axis referred to and the line from the origin to (x, y, z), and λ the cosine of the angle between the axes, we have

Axial Differentiation.

$$\frac{dr}{dh_j} = \mu_j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Again if the suffix k indicate another axis

$$\frac{d\mu_j}{dh_k} = \frac{d}{dh_k}\left(l_j\frac{x}{r} + m_j\frac{y}{r} + n_j\frac{z}{r}\right)$$

$$= \frac{1}{r}\left\{\left(l_jl_k + m_jm_k + n_jn_k\right) - \left(l_j\frac{x}{r} + m_j\frac{y}{r} + n_j\frac{z}{r}\right)\frac{dr}{dh_k}\right\}$$

$$= \frac{1}{r}(\lambda_{jk} - \mu_j\mu_k) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Now let two doublets of moments $-\Phi_1$, $+\Phi_1$, with axes parallel to h_1, be placed with their centres on another axis h_2 at distances $-\frac{1}{2}\partial h_2$, $+\frac{1}{2}\partial h_2$ from the origin, the potential at (x, y, z) due to the pair of doublets is

$$V_2 = -V_1 + V_1 - \partial h_2\frac{dV_1}{dh_2}$$

$$= -\Phi_1\partial h_2\frac{d}{dh_2}\left(\frac{\mu_1}{r^2}\right).$$

If we diminish $\frac{1}{2}\partial h_2$ indefinitely and increase Φ_1 so that $\Phi_1 \partial h_2$ remains a finite quantity $\Phi_2/2$, we have

$$V_2 = -\Phi_2 \tfrac{1}{2} \frac{d}{dh_2}\left(\frac{\mu_1}{r^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Hence performing the differentiation we get

$$V_2 = -\Phi_2 \tfrac{1}{2}\left(\frac{1}{r^2}\frac{d\mu_1}{dh_2} - \frac{2\mu_1}{r^3}\frac{dr}{dh_2}\right)$$

$$= \tfrac{1}{2}\frac{\Phi_2}{r^3}(3\mu_1\mu_2 - \lambda_{12}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This is the potential due to what may be called a doublet of the second order placed at the origin. It may be written

$$V_2 = (-1)^2 \Phi_2 \tfrac{1}{2} \frac{d}{dh_1}\frac{d}{dh_2}\left(\frac{1}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Let now the doublet of the second order we have just supposed built up, be imagined placed with change of direction with its centre on a third axis h_3 at a distance $\frac{1}{2}\partial h_3$ from the origin, and an equal doublet of the second order but of opposite sign placed with its centre on the same axis at the same distance from the origin on the opposite side. Then the potential of this arrangement at (x, y, z) is

$$V_3 = (-1)^3 \Phi_2 \partial h_3 \frac{1}{1 \,.\, 2} \frac{d}{dh_1}\frac{d}{dh_2}\frac{d}{dh_3}\left(\frac{1}{r}\right)$$

If we diminish ∂h_3 and increase Φ_2 so that $\Phi_2 \partial h_3$ remains finite and equal to $\Phi_3/3$, we get a doublet of the third order at the origin with axes h_1, h_2, h_3, which produces a potential at (x, y, z) of amount

$$V_3 = (-1)^3 \Phi_3 \frac{1}{1 \,.\, 2 \,.\, 3} \frac{d}{dh_1}\frac{d}{dh_2}\frac{d}{dh_3}\left(\frac{1}{r}\right) \quad . \quad . \quad . \quad (19)$$

Axial Definition of Solid Harmonic.

Proceeding in this way we can build up a doublet of any order n with axes h_1, h_2, ... h_n. The potential produced at (x, y, z) by this doublet is

$$V_n = (-1)^n \Phi_n \frac{1}{n\,!} \frac{d}{dh_1}\frac{d}{dh_2} \cdots \frac{d}{dh_n}\left(\frac{1}{r}\right) \quad . \quad . \quad . \quad (20)$$

If $\Phi_n = 1$

$$V_n = (-1)^n \frac{1}{n!} \frac{d}{dh_1} \frac{d}{dh_2} \cdots \frac{d}{dh_n} \left(\frac{1}{r}\right) \quad . \quad . \qquad (20')$$

and is a solid harmonic of degree $-(n+1)$. For, performing the differentiations transforms the equation into

$$V_n = r^{-(n+1)} S_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

where S_n is a function of the n cosines of the angles between the axes, and the line from the origin to (x, y, z) and of the $n(n-1)/2$ cosines between the different pairs of the axes. Also V_n obviously satisfies the definition of a spherical harmonic given above.

The value of S_n can be found by successive applications of (14). Thus

$$\left.\begin{aligned} &S_0 = 1, \qquad S_1 = \mu_1, \qquad S_2 = \tfrac{3}{2}\mu_1\mu_2 - \tfrac{1}{2}\lambda_{12} \\ &S_3 = \tfrac{5}{2}\mu_1\mu_2\mu_3 - \tfrac{1}{2}(\mu_1\lambda_{23} + \mu_2\lambda_{31} + \mu_3\lambda_{12}) \\ &S_4 = \frac{7 \cdot 5}{2 \cdot 4}\mu_1\mu_2\mu_3\mu_4 - \frac{5}{2 \cdot 4}(\mu_1\mu_2\lambda_{34} + \mu_2\mu_3\lambda_{41} + \mu_3\mu_4\lambda_{12} \\ &+ \mu_4\mu_1\lambda_{23} + \mu_1\mu_3\lambda_{24} + \mu_2\mu_4\lambda_{13}) + \frac{1}{2 \cdot 4}(\lambda_{12}\lambda_{34} + \lambda_{23}\lambda_{14} + \lambda_{31}\lambda_{24}) \end{aligned}\right\} \quad (22)$$

The general surface harmonic has the expression [Maxwell, *El. and Mag.* Vol. I. p. 188, 2nd Ed.] Expression of General Surface Harmonic.

$$S_n = \sum \left[(-1)^s \frac{(2n-2s)!}{2^{n-s}\, n!\, (n-s)!} \sum (\mu^{n-2s} \lambda^s) \right] \quad . \qquad (23)$$

in which $\Sigma(\mu^{n-2s}\lambda^s)$ denotes the sum of all products of terms of which s of the factors are different cosines λ with double suffixes and $n - 2s$ factors are different cosines μ with single suffixes, and the external Σ denotes summation for all values of s from 0 to $\frac{1}{2}n$. It is clear, since the suffix of each axis appears once and once only in each term, being brought in by the differentiation with respect to that axis, that if there be s factors with double suffixes in any term there must be $n - 2s$ factors in the same term with single suffixes.

Derivation of Zonal Surface Harmonics.

If all the axes coincide, say with the axis of z the harmonic becomes a zonal solid harmonic and S_n degenerates into a surface harmonic of order n. Thus the solid harmonic is

$$r^{-(n+1)} Z_n = (-1)^n \frac{1}{n!} \frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right) \quad . \quad . \quad . \quad . \quad (24)$$

and

$$Z_n = (-1)^n \frac{r^{n+1}}{n!} \frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right) \quad . \quad . \quad . \quad (25)$$

It may be verified by expansion that this agrees with (11) and (12).

Proof Fundamental Relations.

The fundamental relations used at pp. 47, 273 can be deduced from equations (8) and (12). We shall take the first of (80) p. 47, and (50) p. 273, as examples. By (12)

$$2^n n!\, Z'_n = \frac{d^{n+1}}{d\mu^{n+1}}\{(\mu^2-1)^n\}$$

$$= 2n \frac{d^n}{d\mu^n}\{\mu(\mu^2-1)^{n-1}\} \quad . \quad . \quad . \quad . \quad (26)$$

But if u denote any function of μ we have by successive differentiation

$$\frac{d^n}{d\mu^n}(\mu u) = n\frac{d^{n-1}u}{d\mu^{n-1}} + \mu\frac{d^n u}{d\mu^n} \quad . \quad . \quad . \quad . \quad (27)$$

Putting $u = (\mu^2-1)^{n-1}$, and using this result in (26) multiplied by $\mu^2 - 1$, we get

$$2^{n-1}(n-1)!\,(\mu^2-1)\, Z'_n = n\mu^2 \frac{d^{n-1}}{d\mu^{n-1}}\{(\mu^2-1)^{n-1}\}$$

$$+ \mu(\mu^2-1)\frac{d^n}{d\mu^n}\{(\mu^2-1)^{n-1}\} - n\frac{d^{n-1}}{d\mu^{n-1}}\{(\mu^2-1)^{n-1}\} \quad . \quad (28)$$

But by (8)

$$\mu(\mu^2-1)\frac{d^n}{d\mu^n}\{(\mu^2-1)^{n-1}\} = -\mu n(n-1)\int_\mu^1 \frac{d^{n-1}}{d\mu^{n-1}}\{(\mu^2-1)^{n-1}\}d\mu$$

$$= \mu n(n-1)\frac{d^{n-2}}{d\mu^{n-2}}\{(\mu^2-1)^{n-1}\}$$

since the integral vanishes at the superior limit. Hence, taking the two first terms on the right of (28), we get

$$n\mu^2 \frac{d^{n-1}}{d\mu^{n-1}}\{(\mu^2-1)^{n-1}\} + \mu(\mu^2-1)\frac{d^n}{d\mu^n}\{(\mu^2-1)^{n-1}\}$$

$$= n\mu\left[(n-1)\frac{d^{n-2}}{d\mu^{n-2}}\{(\mu^2-1)^{n-1}\} + \mu\frac{d^{n-1}}{d\mu^{n-1}}\{(\mu^2-1)^{n-1}\}\right]$$

$$= n\mu\frac{d^{n-1}}{d\mu^{n-1}}\{\mu(\mu^2-1)^{n-1}\} \quad \text{[by (27)]}$$

$$= \frac{n}{2n}\mu\frac{d^n}{d\mu^n}\{(\mu^2-1)^n\}$$

$$= 2^{n-1}(n-1)!\,n\mu Z_n.$$

Substituting in (28) and dividing by $2^{n-1}(n-1)!$ we find

$$(\mu^2-1)Z'_n = n\mu Z_n - nZ_{n-1} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

which is the first of (30), and the first of the two relations used at p. 273 to obtain (50) and (50′). We can still more easily prove (50), p. 273, directly; we have

$$Z'_n = \frac{1}{2^n n!}\frac{d^{n+1}}{d\mu^{n+1}}\{(\mu^2-1)^n\}$$

$$= \frac{2n}{2^n n!}\frac{d^n}{d\mu^n}\{\mu(\mu^2-1)^{n-1}\}$$

$$= \frac{1}{2^{n-1}(n-1)!}\left[n\frac{d^{n-1}}{d\mu^{n-1}}\{(\mu^2-1)^{n-1}\} + \mu\frac{d^n}{d\mu^n}\{(\mu^2-1)^{n-1}\}\right]$$

[by (27)]

$$= nZ_{n-1} + \mu Z'_{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Hence

$$Z_{n-1} = \frac{1}{n}(Z'_n - \mu Z'_{n-1})$$

which is (50).

The other relations may be established by similar processes.

Theorem of Product of Two Zonal Surface Harmonics

The following theorem is of great importance:—If Z_m, Z_n, be two zonal surface harmonics of orders m, n,

$$\int_{\mu}^{1} Z_m Z_n d\mu = \frac{(1-\mu^2)(Z_m Z'_n - Z_n Z'_m)}{(n-m)(m+n+1)} \quad . \qquad (31)$$

To prove it we have by (8)

$$\frac{d}{d\mu}\{(1-\mu^2)Z'_m\} + m(m+1)Z_m = 0$$

$$\frac{d}{d\mu}\{(1-\mu^2)Z'_n\} + n(n+1)Z_n = 0.$$

Multiplying the first of these by Z_n the second by Z_m and subtracting, observing that $n(n+1) - m(m+1) = (n-m)(m+n+1)$, we find

$$(n-m)(m+n+1)Z_m Z_n = Z_n \frac{d}{d\mu}\{(1-\mu^2)Z'_m\} - Z_m \frac{d}{d\mu}\{(1-\mu^2)Z'_n\}$$

$$= \frac{d}{d\mu}\{(1-\mu^2)(Z_n Z'_m - Z_m Z'_n)\}$$

which gives (31) at once by integration.

If the integral in (31) be taken from -1 to $+1$, then $1-\mu^2 = 0$, at both limits, and the expression on the right vanishes unless either $n = m$, or $n = -(m+1)$. Hence if neither of these conditions is fulfilled

$$\int_{-1}^{+1} Z_m Z_n \, d\mu = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (32)$$

Spherical Harmonic Expansions.

We shall now give some examples of the use of spherical harmonics in expansions. First we shall take the expansion of $1/PP'$ where PP' is the distance of a point P from another point P'. Let r, r', be the distances of the points from the origin, μ the cosine of the angle POP', then we have

$$\frac{1}{PP'} = (r^2 - 2\mu r r' + r'^2)^{-\frac{1}{2}}.$$

If we write h for r'/r, and if $h<1$ we can expand this in a convergent series of ascending powers of h. But we have

seen that Z_n is the coefficient of h^n in the expansion of $(1-2\mu h+h^2)^{-\frac{1}{2}}$. Hence

$$\frac{1}{PP'} = \frac{1}{r}\{Z_0 + Z_1 h + Z_2 h^2 + \ldots\} \quad . \quad . \quad . \quad (33)$$

If $r'/r > 1$ we have only to put $h = r/r'$ and we get

$$\frac{1}{PP'} = \frac{1}{r'}\{Z_0 + Z_1 h + Z_2 h^2 + \ldots\} \quad . \quad . \quad . \quad (33')$$

By means of this result the potential of any distribution whether of attracting matter, or of electricity or magnetism, can be expressed in a series of zonal harmonics.

For let A be the distribution, P' the position of an element, P the point at which the potential is to be found. Then taking

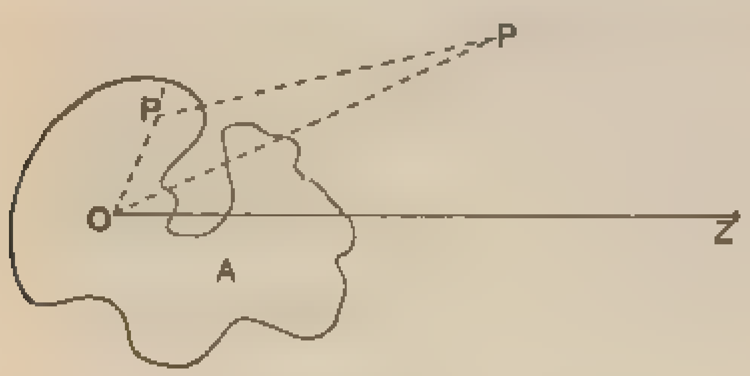

Fig. 198.

coordinates from an origin O, r, r', are the distances OP, OP', and μ the cosine of the angle POP'. Hence if $d\varpi$ is an element of the distribution its potential is

Spherical Harmonic Series for Potential of any given Distribution

$$\frac{d\varpi}{PP'} = \frac{d\varpi}{r}(Z_0 + Z_1 h + Z_2 h^2 + \ldots) \quad . \quad . \quad (34)$$

if $r > r'$, and

$$\frac{d\varpi}{PP'} = \frac{d\varpi}{r'}(Z_0 + Z_1 h + Z_2 h^2 + \ldots) \quad . \quad (34')$$

if $r' > r$.

The total potential is thus

$$V = \int \frac{d\varpi}{r'} (Z_0 + Z_1 h + Z_2 h^2 + \ldots .)$$

or

$$V = \int \frac{d\varpi}{r} (Z_0 + Z_1 h + Z_2 h^2 + \ldots .) \qquad (35)$$

the integral being taken throughout the distribution.

If for one part of the distribution $r > r'$, and for another part $r < r'$, the integration must be divided into two corresponding parts, one for which $h = r/r'$, and the other for which $h = r'/r$.

If ZOP be denoted by θ', ZOP by θ, and the angle which the plane of P' and the axis OZ makes with a fixed plane through the axis by ϕ', then if ρ be the density of the distribution at P'

$$d\varpi = \rho r'^2 \sin\theta' \, d\theta' \, d\phi' \, dr',$$

and the integral must be taken between limits 0 and π for θ' 0 and 2π for ϕ', and 0 and r'_1 for r', where r'_1 is the superior limit of r for given values of θ and ϕ'.

Legendre's Theorem for Distributions Symmetrical about an Axis.

An important theorem due to Legendre greatly facilitates calculations of potentials, forces, &c., for the case of symmetry round an axis. Let it be possible to express the quantity, (supposed to satisfy Laplace's equation) which it is desired to calculate, for points along the axis in a series of ascending or descending powers of z, according as may be necessary for convergence. Thus for points on the axis let the quantity sought be v_a, then by hypothesis

$$v_a = a + \frac{a_0}{z} + \frac{a_1}{z^2} + \frac{a_2}{z^3} + \ldots .$$

or

$$v_a = a'_0 + a'_1 z + a'_2 z^2 + a'_3 z^3 + \ldots . \qquad (36)$$

We can from these expressions find the value of v for any point not on the axis, say at a distance ζ from it. If $r^2 = \sqrt{z^2 + \zeta^2}$ we have

$$v = a + a_0 \frac{Z_0}{r} + a_1 \frac{Z_1}{r^2} + \ldots .$$

or

$$v = a'_0 Z_0 + a'_1 Z_1 r + a'_2 Z_2 r^2 + \ldots . \qquad (37)$$

that is we have only to substitute r for z, and multiply the terms of coefficients a_0, a_1, &c by the zonal surface harmonics of orders indicated by the suffixes. It is to be observed that the zonal surface harmonics are chosen for the terms in the two series, so that in each case the terms are the successive zonal solid harmonics, in the first series of degrees $1, -2, -3$, &c., in the second of degrees 0, 1, 2, 3 &c. These involve in both cases the same successive surface harmonics of orders 0, 1, 2, 3, &c., according to the theorem proved above that to every solid harmonic $r^n S_n$, of degree n, there corresponds another $r^{-(n+1)} S_n$ of degree $-(n+1)$.

As an example take the case of a wire bent into a circle of radius a, and carrying a current γ. The magnetic potential at a point on the axis of the circle at distance z from the centre is

Potential due to Circular Current.

$$V_a = 2\pi\gamma\left(1 - \frac{z}{\sqrt{z^2 + a^2}}\right)$$

We may write $1 - z/\sqrt{z^2 + a^2}$ in the form $1 - (1 + a^2/z^2)^{-\frac{1}{2}}$, and if $a < z$ expand in descending powers of z. Thus we find

$$V_a = 2\pi\gamma\left(\frac{1}{2}\frac{a^2}{z^2} - \frac{3}{8}\frac{a^4}{z^4} + \frac{5}{16}\frac{a^6}{z^6} - \frac{35}{128}\frac{a^8}{z^8} + \ldots\right) \qquad (38)$$

In like manner if $a > z$, we obtain

$$V_a = 2\pi\gamma\left(1 - \frac{z}{a} + \frac{1}{2}\frac{z^3}{a^3} - \frac{3}{8}\frac{z^5}{a^5} + \frac{5}{16}\frac{z^7}{a^7} - \ldots\right) \qquad (39)$$

Thus for points taken anywhere we get from (38) and (39)

$$V = 2\pi\gamma\left(\frac{1}{2}a^2\frac{Z_1}{r^3} - \frac{3}{8}a^4\frac{Z_3}{r^5} + \frac{5}{16}a^6\frac{Z_5}{r^7} - \ldots\right)$$

or

$$V = 2\pi\gamma\left(1 - \frac{r}{a}Z_1 + \frac{1}{2}\frac{r^3}{a^3}Z_3 - \frac{3}{8}\frac{r^5}{a^5}Z_5 + \ldots\right), \qquad (40)$$

according as $a <$ or $> r$.

This is really the problem treated at p. 46 above. Another example is given by the problem of two shells discussed at pp. 48, 49, above.

Proof of Legendre's Theorem

The theorem used in equations (37) and (40) may be regarded as a limiting case of Green's theorem, that if a function of x, y, z is found to satisfy Laplace's equation throughout space external to a closed surface, and to give specified values for points on the surface, that function is the only one fulfilling these conditions. In the present case the closed surface is shrunk into a line, and in strictness the theorem requires special demonstration. Legendre's own proof will be found in Minchin's Statics Vol. II., p. 341 (Sec. Ed.) The following proof given by Minchin p. 324, *loc. cit.*, is simpler.

For the case of symmetry round an axis if ζ be the distance of the point considered from the axis Laplace's equation takes the form

$$\zeta\left(\frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial \zeta^2}\right) + \frac{\partial V}{\partial \zeta} = 0 \quad . \quad . \quad . \quad . \quad (41)$$

which it is to be noted gives

$$\frac{\partial V}{\partial \zeta} = 0, \quad \frac{\partial^2 V}{\partial \zeta^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (42)$$

at all points on the axis in the space throughout which it is supposed that the equation holds.

If then we know a function U which satisfies (41), and gives the specified values at points on the axis, let if possible V be *another* function which does the same thing. Then $V - U (= \Phi$ say) must fulfil (41), and be zero at points on the axis. Hence at all such points

$$\frac{\partial \Phi}{\partial z} = 0, \quad \frac{\partial^2 \Phi}{\partial z^2} = 0, \ldots .$$

We can now show that for any point on the axis

$$\frac{\partial^{m+n} \Phi}{\partial z^m d\zeta^n} = 0$$

For at any point on the axis we have seen [(42)] that $\partial\Phi / \partial\zeta = 0$, $\partial^2\Phi / \partial\zeta^2 = 0$, and by (27) above

$$\frac{\partial^n}{\partial \zeta^n}\left\{\zeta\left(\frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Phi}{\partial \zeta^2}\right) + \frac{\partial \Phi}{\partial \zeta}\right\}$$

$$= n\frac{\partial^{n+1}\Phi}{\partial z^2 \partial \zeta^{n-1}} + (n+1)\frac{\partial^{n+1}\Phi}{\partial \zeta^{n+1}} + \zeta\left(\frac{\partial^{n+2}\Phi}{\partial z^2 \partial \zeta^n} + \frac{\partial^{n+2}\Phi}{\partial \zeta^{n+2}}\right) = 0$$

Hence for points on the axis

$$n\frac{\partial^2}{\partial z^2}\left(\frac{\partial^{n-1}\Phi}{\partial\zeta^{n-1}}\right) + (n+1)\frac{\partial^{n+1}\Phi}{\partial\zeta^{n+1}} = 0.$$

If therefore $\partial^{n-1}\Phi/\partial\zeta^{n-1} = 0$ for points on the axis, $\partial^{n+1}\Phi/\partial\zeta^{n+1} = 0$. But $\partial\Phi/\partial\zeta = 0$, and $\partial^2\Phi/\partial\zeta^2 = 0$, and therefore $\partial^3\Phi/\partial\zeta^3 = 0$, and so on. Hence it follows, since the differentiations are commutative, that $\partial^{m+n}\Phi/\partial z^m\partial\zeta^n = 0$.

Expressing then Φ as $f(z, \zeta)$ and expanding by Maclaurin's theorem, denoting values of Φ, $\partial\Phi/\partial z$, &c., for points at the origin by the suffix 0, we get

$$\Phi = \Phi_0 + z\frac{\partial\Phi}{\partial z_0} + \zeta\frac{\partial\Phi}{\partial\zeta_0} + \frac{1}{1\,.\,2}\left(z^2\frac{\partial^2\Phi}{\partial z^2_0} + 2z\zeta\frac{\partial^2\Phi}{\partial z\partial\zeta} + \zeta^2\frac{\partial^2\Phi}{\partial\zeta^2_0}\right)$$
$$+ \text{\&c.} = 0$$

since all the differential coefficients vanish.

Hence $\Phi = 0$, everywhere, which proves that U cannot differ from V.

It is shown above, p. 274, that

$$(-1)^{i+1}(i-1)\,!\,a^2\int\frac{Z'_i}{r^{i+2}}\,dx = \frac{\partial^i A}{\partial x^i}$$

Calculation of $\int\frac{Z'_i}{r^{i+2}}\,dx$.

where Z_i is a zonal harmonic of order i, $x = \mu r$, and $A = \sqrt{a^2+x^2} - x$. The evaluation of these integrals is of great importance for the calculation of the inductances of coils, and by this theorem they can be obtained at once by simply finding the successive differential coefficients of A. As promised we give here the first eleven differential coefficients. It may be noted that they can be written down with great facility from the known expressions for the successive zonal harmonics by the equation

$$(-1)^{i+1}(i-1)\,!\,a^2\frac{Z'_i}{r^{i+2}} = \frac{\partial^{i+1}A}{\partial x^{i+1}}$$

$$\frac{\partial A}{\partial x} = \frac{x}{r} - 1, \qquad \frac{\partial^2 A}{\partial x^2} = \frac{a^2}{r^3}.$$

$$\frac{\partial^3 A}{\partial x^3} = -\frac{3a^2x}{r^5}, \qquad \frac{\partial^4 A}{\partial x^4} = 3a^2(5x^2 - r^2)\frac{1}{r^7}.$$

$$\frac{\partial^5 A}{\partial x^5} = -\,3\,.\,5a^2x(7x^2 - 3r^2)\,\frac{1}{r^9}.$$

$$\frac{\partial^6 A}{\partial x^6} = 3^2\,.\,5a^2(21x^4 - 14x^2r^2 + r^4)\,\frac{1}{r^{11}}.$$

$$\frac{\partial^7 A}{\partial x^7} = -\,3^2\,.\,5a^2x(231x^4 - 210x^2r^2 + 35r^4)\,\frac{1}{r^{13}}.$$

$$\frac{\partial^8 A}{\partial x^8} = 3^2\,.\,5a^2(3003x^6 - 3465x^4r^2 + 945x^2r^4 - 35r^6)\,\frac{1}{r^{15}}.$$

$$\frac{\partial^9 A}{\partial x^9} = -\,3^2\,.\,5\,.\,7a^2x(6435x^6 - 9009x^4r^2 + 3465x^2r^4 - 315r^6)\,\frac{1}{r^{17}}$$

$$\frac{\partial^{10} A}{\partial x^{10}} = 3^3\,.\,5^2\,.\,7a^2(7293x^8 - 12012x^6r^2 + 6006x^4r^4 - 924x^2r^6 + 21r^8)\,\frac{1}{r^{19}}.$$

$$\frac{\partial^{11} A}{\partial x^{11}} = -\,3^2\,.\,5^2\,.\,7\,.\,9a^2x(46189x^8 - 87516x^6r^2 + 54054x^4r^4 - 12012x^2r^6 + 693r^8)\,\frac{1}{r^{21}}.$$

TABLE OF ZONAL SPHERICAL HARMONICS

(Prof. Perry, *Phil. Mag.* Dec. 1891. See also p. 824 above).

θ.	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7
0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
1	·9998	·9995	·9991	·9985	·9977	·9967	·9955
2	·9994	·9982	·9963	·9939	·9909	·9872	·9829
3	·9986	·9959	·9918	·9863	·9795	·9713	·9617
4	·9976	·9927	·9854	·9758	·9638	·9495	·9329
5	·9962	·9886	·9773	·9623	·9437	·9216	·8961
6	·9945	·9836	·9674	·9459	·9194	·8881	·8522
7	·9925	·9777	·9557	·9267	·8911	·8470	·7986
8	·9903	·9709	·9423	·9048	·8589	·8053	·7448
9	·9877	·9633	·9273	·8803	·8232	·7571	·6831
10	·9848	·9548	·9106	·8532	·7840	·7045	·6164
11	·9816	·9454	·8923	·8238	·7417	·6483	·5461
12	·9781	·9352	·8724	·7920	·6966	·5892	·4732
13	·9744	·9241	·8511	·7582	·6489	·5273	·3940
14	·9703	·9122	·8283	·7224	·5990	·4635	·3219
15	·9659	·8995	·8042	·6847	·5471	·3982	·2454
16	·9613	·8860	·7787	·6454	·4937	·3322	·1699
17	·9563	·8718	·7519	·6046	·4391	·2660	·0961
18	·9511	·8568	·7240	·5624	·3836	·2002	·0289
19	·9455	·8410	·6950	·5192	·3276	·1347	− ·0443
20	·9397	·8245	·6649	·4750	·2715	·0719	·1072
21	·9336	·8074	·6338	·4300	·2156	·0107	·1662
22	·9272	·7895	·6019	·3845	·1602	·0481	·2201
23	·9205	·7710	·5692	·3386	·1057	·1038	− ·2681
24	·9135	·7518	·5357	·2926	·0525	− ·1559	− ·3095
25	·9063	·7321	·5016	·2465	·0009	·2053	·3463
26	·8988	·7117	·4670	·2007	− ·0489	·2478	·3717
27	·8910	·6908	·4319	·1558	− ·0964	·2869	·3921
28	·8829	·6694	·3964	·1105	·1415	− ·3211	·4052
29	·8746	·6474	·3607	·0665	·1839	·3503	·4114
30	·8660	·6250	·3248	·0234	− ·2233	·3740	·4101
31	·8572	·6021	·2887	·0185	·2595	·3924	·4022
32	·8480	·5788	·2527	·0591	− ·2923	·4052	− ·3876
33	·8387	·5551	·2167	− ·0982	− ·3216	− ·4126	− ·3670
34	·8290	·5310	·1809	− ·1357	·3473	− ·4148	·3409
35	·8192	·5065	·1454	− ·1714	·3691	·4115	·3096
36	·8090	·4818	·1102	− ·2052	·3871	− ·4031	·2738
37	·7986	·4567	·0756	− ·2370	·4011	·3898	·2348
38	·7880	·4314	·0418	− ·2666	− ·4112	·3719	− ·1918
39	·7771	·4059	·0077	·2940	·4174	− ·3497	·1469
40	·7660	·3802	− ·0252	·3190	·4197	− ·3234	·1003
41	·7547	·3544	·0574	·3416	·4181	− ·2938	·0534
42	·7431	·3284	·0887	·3616	·4128	·2611	·0065
43	·7314	·3023	·1191	·3791	− ·4038	·2255	·0398
44	·7193	·2762	− ·1485	− ·3940	− ·3914	·1878	·0846
45	·7071	·2500	− ·1768	·4062	·3757	− ·1485	·1270

TABLE (*continued*)

θ.	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7
46	·6947	·2238	·2040	− ·4158	·3468	+ ·1079	·1666
47	·6820	·1977	·2300	·4252	·3350	− ·0684	·2054
48	·6691	·1716	·2547	− ·4270	− ·3163	− ·0241	·2349
49	·6561	·1459	− ·2781	− ·4286	− ·2936	+ ·0161	·2627
50	·6428	·1198	·3002	− ·4275	·2545	+ ·0563	·2854
51	·6293	·0941	·3209	·4239	·2335	+ ·0954	·3031
52	·6157	·0684	·3401	·4178	·1910	+ ·1329	·3153
53	·6018	·0429	·3578	− ·4093	− ·1471	+ ·1677	·3221
54	·5878	·0187	·3740	·3984	·1223	+ ·2002	·3234
55	·5736	·0065	·3880	− ·3852	·0968	+ ·2297	·3191
56	·5592	·0310	·4016	·3698	·0510	+ ·2559	·3095
57	·5446	− ·0551	·4131	·3524	− ·0150	+ ·2787	·2949
58	·5299	·0788	− ·4229	− ·3332	·0206	+ ·2976	·2752
59	·5150	·1021	·4310	·3119	·0557	+ ·3125	·2511
60	·5000	− ·1250	·4375	− ·2891	·0898	+ ·3232	·2231
61	·4848	− ·1474	·4423	− ·2647	·1229	+ ·3298	·1916
62	·4695	·1694	·4455	− ·2390	·1545	+ ·3321	·1571
63	·4540	·1908	− ·4471	·2121	·1844	+ ·3302	·1208
64	·4384	·2117	− ·4470	·1841	·2129	+ ·3240	·0818
65	·4226	·2321	·4452	·1552	·2381	+ ·3138	·0422
66	·4067	·2518	− ·4419	·1256	·2615	+ ·2995	·0021
67	·3907	− ·2710	·4370	·0955	·2824	+ ·2819	·0474
68	·3746	·2896	− ·4305	− ·0650	·3005	+ ·2605	·0703
69	·3584	− ·3074	− ·4225	− ·0344	·3158	+ ·2361	·1135
70	·3420	− ·3244	·4130	·0038	·3281	+ ·2089	·1487
71	·3256	− ·3410	·4021	·0267	·3373	+ ·1786	− ·1811
72	·3090	·3568	·3898	·0568	·3434	+ ·1472	·2099
73	·2924	·3718	− ·3761	·0864	·3463	+ ·1144	·2347
74	·2756	− ·3860	− ·3611	·1153	·3461	+ ·0795	− ·2559
75	·2588	− ·3995	·3449	·1434	·3427	+ ·0431	·2730
76	·2419	·4119	·3275	·1705	·3362	+ ·0076	− ·2848
77	·2250	·4241	·3090	·1964	·3267	·0294	·2919
78	·2079	− ·4352	− ·2894	·2211	·3143	·0644	− ·2943
79	·1908	·4454	·2688	·2443	·2990	− ·0989	·2918
80	·1736	− ·4548	− ·2474	·2659	·2810	·1321	·2835
81	·1564	·4633	·2251	·2850	·2606	− ·1635	− ·2700
82	·1392	− ·4709	− ·2020	·3040	·2378	·1926	·2536
83	·1219	− ·4777	− ·1783	·3203	·2129	·2193	− ·2321
84	·1045	·4836	− ·1539	·3344	·1861	− ·2431	− ·2067
85	·0872	− ·4886	− ·1291	·3463	·1577	·2638	·1779
86	·0698	·4927	− ·1036	·3559	·1278	− ·2811	− ·1460
87	·0523	− ·4959	− ·0781	·3648	·0969	·2947	·1117
88	·0349	·4982	·0522	·3704	·0651	− ·3045	− ·0735
89	·0175	·4995	·0262	·3739	·0327	− ·3105	− ·0381
90	·0000	− ·5000	− ·0000	·3750	·0000	·3125	·0000

II.

ERRORS OF OBSERVATION AND THE COMBINATION OF EXPERIMENTAL RESULTS.

ALL observations of physical quantities are subject to two kinds of errors, (1) constant and therefore avoidable errors, (2) errors which are due to the inherent inaccuracy of observation, and which may be said to be accidental.

Nature of Errors of Observation

Constant and Accidental Errors.

Among the former are errors due to some cause which affects all the operations of a certain class to the same extent, for example, a constant wind blowing across a rifle-range, the personal equation of an observer, or the zero-error of the scale of an instrument. The latter comprise errors such as those produced in striking a target by inevitable inaccuracies of aim, errors in reading the scale of an instrument through inaccuracy of setting or of estimation of fractions of a division, &c.

The former class of errors can in general be very exactly eliminated from all observations, and we shall not discuss them. The latter class being regulated by no one definite physical cause are as liable to be errors of defect as of excess, that is, positive errors are as probable in the mathematical sense as are negative errors. By this we mean that in a large number of observations of a quantity, the true value of which is accurately known, the differences between the true value and the observed values would be fairly equally distributed on the two sides of the former. Further in all such cases it is matter of common observation that errors occur with less frequency the greater their magnitude, and that very large errors hardly occur at all.

Distribution of Accidental Errors.

Experience shows in fact that accidental errors of observation are distributed according to a certain law, which may be deduced by an application of the theory of probabilities, in the following manner.* We assume that the probability of a negative error is equal to that of a positive error of the same magnitude. Hence the probability of an error of magnitude x must be an even function of x. Thus the probability of an error between x and $x + dx$ (or, briefly, *of an error* x) is $\phi(x^2)dx$. But the error lies between $-\infty$ and $+\infty$, and hence

Error Function.

$$\int_{-\infty}^{+\infty} \phi(x^2)dx = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

* It is assumed that the reader is acquainted with the elementary ideas of mathematical probability. See any treatise on Algebra.

Form of Error Function. The form of the function ϕ may be found in the following manner. Let two axes, one vertical, the other horizontal be ruled through the middle of the bull's eye of a target. The chance that a shot will strike at a distance between x and $x + dx$ from the vertical axis is $\phi(x^2)dx$, and that it will strike at a distance between y and $y + dy$ from the horizontal axis is $\phi(y^2)dy$. Hence the probability that the shot will strike at a point fulfilling both conditions is $\phi(x^2)\phi(y^2)dxdy$. This is the probability that the shot will strike the small area $dxdy$. But this must be the same for an equal small area at the same place whatever pair of rectangular axes through the centre of the target are chosen. If therefore x', y' be coordinates of the point (x, y) when referred to another pair of axes we must have

$$\phi(x'^2)\,\phi(y'^2) = \phi(x^2)\,\phi\,(y^2).$$

This is a functional equation of which the solution is

$$\phi(x^2) = Ae^{mx^2}$$

where A and m are constants. The value of m must be negative as the probability of a large error is very small. Writing therefore $m = -h^2$ we find by (1)

$$\int_{-\infty}^{+\infty}\phi(x^2)dx = A\int_{-\infty}^{+\infty}e^{-h^2x^2}dx = 1$$

Hence by the well-known theorem that

$$\int_0^{\infty}e^{-h^2x^2}dx = \frac{\sqrt{\pi}}{2h}$$

we have $A = h/\sqrt{\pi}$. Thus

$$\phi(x^2) = \frac{h}{\sqrt{\pi}}e^{-h^2x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Measure of Precision. The quantity h is called the measure of precision of the observations For take two errors of equal probability in two different sets of observations for which this constant has the values h, h', we have

$$\int_0^{hx}e^{-h^2x^2}d(hx) = \int_0^{h'x'}e^{-h'^2x'^2}d(h'x')$$

Thus the two probabilities are equal if $hx = h'x'$, that is the errors are inversely proportional to h, h'.

Probability of an error

The probability P that an error lies between x and x' is therefore given by

$$P = \frac{h}{\sqrt{\pi}}\int_x^{x'} \epsilon^{-h^2x^2}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad ($$

Probability Integral.

The probability that an error lies between 0 and x, or

$$\frac{h}{\sqrt{\pi}}\int_0^x \epsilon^{-h^2x^2}dx$$

is called the probability integral, and tables of its values are given in treatises on *Errors of Observation.*

Combination of Results of Observation.

We shall now apply this theory to the reduction of the results of observation. Generally speaking the quantities, the values of which are sought, $x_1, x_2, \ldots, x_m$ (m in number) say, are not those directly observed, but are connected with the latter by known relations or *observational equations,* just as many in number as there are observations. If the observations were perfectly accurate and were $n(> m)$ in number the values of $x_1, x_2, \ldots . x_m$, could be found from any m of them; but as inaccuracy cannot be avoided, a much larger number n of observations is generally made than there are of quantities to be determined, and the n *observational equations* which these give are reduced to m by some process of combination. That usually adopted is derived as follows:—

Let the observed quantities be $M_1, M_2, \ldots, M_n$ and

$$f_1(x_1, x_2 \ldots x_m), f_2(x_1, x_2, \ldots, x_m), \ldots . f_n(x_1, x_2, \ldots x_m)$$

be the *true* values of the quantities observed in terms of those $(x_1, x_2, \ldots, x_m)$ sought. The errors are

$$\left.\begin{array}{l} f_1(x_1, x_2, \ldots, x_m) - M_1 = e_1 \\ f_2(x_1, x_2, \ldots, x_m) - M_2 = e_2 \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ f_n(x_1, x_2, \ldots, x_m) - M_n = e_n \end{array}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

Method of Least Squares.

Now we can never know in any case what the errors actually are. We may however inquire what is the most probable system of errors, and find accordingly $x_1, x_2, \ldots . x_n$ from the observed values of $M_1, M_2, \ldots, M_n$. If the observations are all of the same degree of precision, that is, are made with equally

precise instruments and with the same degree of care and precaution in every case, the probabilities of the several errors e_1, e_2, ..., e_n are proportional to

$$h/\sqrt{\pi} \cdot \epsilon^{-h^2e_1^2},\ h/\sqrt{\pi} \cdot \epsilon^{-h^2e_2^2}, \ldots h/\sqrt{\pi} \cdot \epsilon^{-h^2e_n^2},$$

and the probability of this system of errors to the product of these several probabilities or to $h^n\pi^{-n/2}\epsilon^{-h^2\Sigma e^2}$.

This has its maximum value when Σe^2 or $e_1^2 + e_2^2 + \ldots + e_n^2$, regarded as variable, is a minimum. The most probable system of values of $x_1, x_2, \ldots, x_m$ is therefore that deduced from those of $f_1, f_2, \ldots, f_n$, which fulfil this condition. We shall now show how this condition enables the n observational equations to be combined in m equations in $x_1, x_2, \ldots, x_m$. These are generally called the normal equations.

Since the system of errors of greatest probability is adopted instead of the actual system, we shall now denote the errors by $v_1, v_2, \ldots, v_n$, and call them *residuals*.

Formation of Normal Equations. Squaring both sides of equations (4) and adding we obtain

$$\Sigma\{f(x_1, x_2, \ldots, x_m) - M\}^2 = v_1^2 + v_2^2 + \ldots + v_n^2$$

the quantity on the left is to be made a minimum. Denoting it by S we find as the necessary condition

$$\frac{\partial S}{\partial x_1} = 0,\ \frac{\partial S}{\partial x_2} = 0, \ldots\ldots, \frac{\partial S}{\partial x_m} = 0 \quad . \quad . \quad . \quad . \quad (5)$$

which are the m equations required to find $x_1, x_2, \ldots, x_m$.

When the relations $f_1, f_2, \ldots, f_n$, are all linear an easy rule is obtained. For we have then instead of (4)

$$\left.\begin{array}{l} \alpha_1x_1 + \beta_1x_2 + \ldots\ldots + \mu_1x_m - M_1 = v_1 \\ \alpha_2x_1 + \beta_2x_2 + \ldots\ldots + \mu_2x_m - M_2 = v_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_nx_1 + \beta_nx_2 + \ldots\ldots + \mu_nx_m - M_n = v_n \end{array}\right\} \quad (6)$$

and equations (5) become

$$\begin{aligned} &\alpha_1(\alpha_1x_1 + \beta_1x_2 + \ldots + \mu_1x_m) + \alpha_2(\alpha_2x_1 + \beta_2x_2 + \ldots + \mu_2x_m) \\ &\qquad + \ldots\ldots + \alpha_n(\alpha_nx_1 + \beta_nx_2 + \ldots + \mu_nx_m) \\ &\qquad\qquad = \alpha_1M_1 + \alpha_2M_2 + \ldots + \alpha_nM_n \\ &\beta_1(\alpha_1x_1 + \beta_1x_2 + \ldots + \mu_1x_m) + \beta_2(\alpha_2x_1 + \beta_2x_2 + \ldots + \mu_2x_m) \qquad (7) \\ &\qquad + \ldots\ldots + \beta_n(\alpha_nx_1 + \beta_nx_2 + \ldots + \mu_nx_m) \\ &\qquad\qquad = \beta_1M_1 + \beta_2M_2 + \ldots + \beta_nM_n \end{aligned}$$

. . .

Thus we have the rule to form the normal equations for $x_1, x_2, \ldots, x_m$: *multiply the left-hand member of each equation of* (6) *by the coefficient of x_1 in that equation, add the results together, and equate to zero for the first equation; then multiply the left-hand member of each equation of* (6) *by the coefficient of x_2 in the same, add the results together, and equate to zero for the second equation; and so on.*

Example: Observations of Period of Vibration.

As an example we take the case of two unknown quantities, x_1, x_2, connected with the observations by the equations

$$\left.\begin{aligned} a_1x_1 + x_2 - M_1 &= v_1 \\ a_2x_1 + x_2 - M_2 &= v_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \\ a_nx_1 + x_2 - M_n &= v_n \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Then forming the normal equations and solving for x_1, x_2, we get

$$\left.\begin{aligned} x_1 &= \frac{\Sigma a \,.\, \Sigma M - n\Sigma(aM)}{(\Sigma a)^2 - n\Sigma(a^2)} \\ x_2 &= \frac{\Sigma a \,.\, \Sigma(aM) - \Sigma(a^2) \,.\, \Sigma M}{(\Sigma a)^2 - n\Sigma a^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

If, as in many physical experiments, $a_1 = 0$, and $a_2, a_3, \ldots a_n$, be in arithmetic progression, the equation for x_1 leads at once to the rules of combination given at pp. 84, 85 above in connection with observations of the period of vibration of a magnet.

Direct Observations of a Single Quantity.

It is easy to show that this process gives as the most probable value of a single quantity n times directly observed, the arithmetic mean of the n observed values. For we have only to put f for the quantity to be found, and $M_1, M_2, \ldots, M_n$ for the observed values. We get then

$$(f - M_1)^2 + (f - M_2)^2 + \ldots + (f - M_n)^2 = \text{a minimum},$$

that is,

$$f - M_1 + f - M_2 + \ldots + f - M_n = 0$$

or

$$f = \frac{M_1 + M_2 + \ldots + M_n}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The Arithmetic Mean the Most Probable Value.

Thus the most probable value of f is the arithmetic mean of $M_1, M_2, \ldots M_n$. The ordinary simple rule used in this case therefore coincides with that given by the present theory.

Weights of Observations.

When results of observation are of different degrees of precision they are "weighted" by being each multiplied by a factor which depends upon the degree of precision which the

observations are supposed to have. For example, if $M_1, M_2, \ldots M_n$, in the last example were of different degrees of precision, they would have to be weighted by being multiplied respectively by weights $w_1, w_2, \ldots w_n$, and we should have instead of (10)

$$f = \frac{w_1 M_1 + w_2 M_2 + \ldots + w_n M_n}{w_1 + w_2 + \ldots + w_n} \quad . \quad . \quad . \quad (11)$$

This is called the general mean.

Probable Error. The method generally resorted to in assigning weights to observations depends on the determination of what is called the *probable error* of each result. This is the error of probability $\frac{1}{2}$, that is to say, it is just as probable that the error of the result is less than this as that it is greater. Thus the probable error is the value of r in the equation

$$\tfrac{1}{2} = \frac{2h}{\sqrt{\pi}} \int_0^r e^{-h^2 u^2} du.$$

This equation can be solved by interpolation from a table of values of the probability integral and gives

$$r = \frac{\cdot 4769}{h}.$$

The probable error is thus inversely proportional to h.

Series of Observations on One Quantity. When a series of observations n in number is made, the probable error P of the result is found in the following manner. The probability of the system of errors $e_1, e_2 \ldots e_n$ is

$$P = \frac{h^n}{\pi^{n/2}} (dx)^n e^{-h^2(e_1^2 + e_2^2 + \ldots + e_n^2)} \quad . \quad . \quad . \quad (11')$$

Now if the observations have been made with all possible care so as to have as great a degree of precision as was possible, that is, to have as small a probable error as possible consistent with the given system of errors, we must assign to h such a value as to make P in (8) a maximum. Differentiating P with respect to h and equating to zero, we find

Probable Error of Single Observation

$$\frac{nP}{h} - 2h\Sigma(e^2) \, . \, P = 0$$

or

$$h = \sqrt{\frac{n}{2\Sigma(e^2)}}$$

But $h = \cdot 4769\, r$, and therefore

$$r = \cdot 6745 \sqrt{\frac{\Sigma(e^2)}{n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$\Sigma(e^2)$, the sum of the squares of the true errors, is greater than $\Sigma(v^2)$, the sum of the squares of the residual errors, and the difference is less the greater the number of observations. As an approximation it is usual to put

$$\Sigma(v^2) = \Sigma(e^2)\left(1 - \frac{1}{n}\right).$$

Thus

$$r = \cdot 6745 \sqrt{\frac{\Sigma(v^2)}{n-1}} \quad . \quad . \quad . \quad . \quad (13)$$

This is called the *probable error* of a single observation of a single quantity directly observed. The quantity $\sqrt{\Sigma(v^2)/(n-1)}$ is called the *mean error.*

Probable Error of Arithmetic Mean.

If $M_1, M_2, \ldots, M_n$ be n obtained values of the same quantity, the probable error of the arithmetic mean can be found. For if we take $z_1, z_2, \ldots, z_n$ as the differences between the observed quantities and the arithmetic mean, the probability of the arithmetic mean is the product, P, of the probabilities of this system of residual errors. Thus we have

$$P = h^n(dz)^n\pi^{-n/2}e^{-h^2(z_1^2 + z_2^2 + \ldots + z_n^2)}.$$

Now let some other quantity differing from the arithmetic mean by an amount δ be taken as the quantity sought. Then instead of $z_1, z_2, \ldots, z_n$ we shall have for residual errors $z_1 + \delta, z_2 + \delta, \ldots, z_n + \delta$. Thus for the probability of this system of residuals we get

$$P' = h^n(dz)^n\pi^{-\frac{n}{2}}e^{-h^2(z_1^2 + z_2^2 + \ldots + z_n^2 + n\delta^2)},$$

since $2(z_1 + z_2 + \ldots + z_n)\delta = 0$. Thus

$$\frac{P}{P'} = \frac{1}{e^{-nh^2\delta^2}}.$$

This states that the probability of the arithmetic mean is to that of the other system of errors, or the probabilities of errors zero and δ in the arithmetic mean are, as $1 : e^{-nh^2\delta^2}$. The probability of an error zero in a single observation is to that of an

error δ as $1 : \epsilon^{-h^2\delta^2}$, and hence the measure of precision in the case of the arithmetic mean is $\sqrt{n}$ times that of a single observation. Hence

$$\text{Probable error of arith. mean} = \frac{r}{\sqrt{n}} = \cdot 6745 \sqrt{\frac{\Sigma(v^2)}{n(n-1)}} \,. \quad (14)$$

Assignment of Weights.

We have no space to deal with the question of the weights to be assigned when results of observations or of different sets of observations, of different degrees of precision, are to be combined. The general rule is to assign to the results weights inversely proportional to the squares of the probable errors of the quantities. These weights are obtained by the following rule for the quantities $x_1, x_2, \ldots x_m$ of equations (7). Denote the quantities on the right of the successive normal equations by $A_1, A_2, \ldots A_m$ respectively, and let these equations solved for $x_1, x_2, \ldots x_m$ give

$$\left.\begin{array}{l} x_1 = a_1A_1 + a_2A_2 + \ldots + a_nA_n \\ x_2 = b_1A_1 + b_2A_2 + \ldots + b_nA_n \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\}$$

Then

$$\text{weight of } x_1 = \frac{1}{a_1},\ \text{weight of } x_2 = \frac{1}{b_2}, \ldots \quad (15)$$

Mean and Probable Errors of Quantities Indirectly Obtained.

The mean errors of $x_1, x_2, \ldots, x_m$ are equal each to the mean error of a single observation divided by the square root of the weight of the quantity in question. But the mean error of a single observation in this case may be shown to be

$$\sqrt{\frac{\Sigma(v_2)}{n-m}} \,.$$

Hence we obtain

$$\left.\begin{array}{l} \text{Mean error of } x_1 = \sqrt{\dfrac{a_1\Sigma(v^2)}{n-m}} \\ \quad ,, \qquad ,, \qquad x_2 = \sqrt{\dfrac{b_2\Sigma(v^2)}{n-m}} \end{array}\right\} \quad \ldots \quad (16)$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

From these the probable errors of $x_1, x_2, \ldots$ are obtained by multiplying by the factor $\cdot 6745$.

As an example the following case (due to Gauss) of four observational equations for three quantities may be taken. [Of course the theory gives good results only when there is a large number of observational equations, but the present example shows very well the mode of proceeding.] Example.

$$\begin{aligned} x \quad - y + 2z - 3 &= 0 \\ 3x + 2y - 5z - 5 &= 0, \\ 4x + y + 4z - 21 &= 0, \\ -x + 3y + 3z - 14 &= 0. \end{aligned}$$

The normal equations are

$$\begin{aligned} 27x + 6y \qquad - 88 &= 0, \\ 6x + 15y + z - 70 &= 0, \\ y + 54z - 107 &= 0, \end{aligned}$$

from which the most probable values of x, y, z, are obtained, *viz.*

$$x = \frac{49154}{19899} = 2{\cdot}470,\ y = \frac{2617}{737} = 3{\cdot}351,\ z = \frac{12707}{6633} = 1{\cdot}916.$$

The weights of x, y, z, determined as described above are respectively

$$\frac{19899}{809} = 24{\cdot}597,\ \frac{737}{54} = 13{\cdot}648,\ \frac{2211}{41} = 53{\cdot}927.$$

The residual errors given by these values of x, y, z, are given in the following table :—

No	v	v^2
1	− 0·249	0·0620
2	− 0·068	0·0046
3	+ 0·095	0·0090
4	0·069	0·0048
		$\Sigma(v^2) = {\cdot}0804$

Here $n = 4$, $m = 3$, and

$$\text{Mean error of an observation} = \sqrt{\frac{\cdot 0804}{4-3}} = \cdot 284.$$

$$\text{Mean error of } x = \frac{\cdot 284}{\sqrt{24 \cdot 597}} = \cdot 057.$$

$$\text{,, \quad ,, \quad } y = \frac{\cdot 284}{\sqrt{13 \cdot 648}} = \cdot 077.$$

$$\text{,, \quad ,, \quad } z = \frac{\cdot 284}{\sqrt{53 \cdot 927}} = \cdot 039.$$

Application of *Least Squares* to Harmonic Analysis.

The method of least squares can be applied to determine the coefficients A_0, A_1, A_2, . . . B_1, B_2, . . . in the equation

$$\left. \begin{aligned} y = A_0 + A_1 \sin nx + A_2 \sin 2nx + \dots \\ + B_1 \cos nx + B_2 \cos 2nx + \dots \end{aligned} \right\} \quad . \quad . \quad (16)$$

when a number of values of y for different known values of x have been determined. In general n is known and therefore the observational equations are at once obtained by substituting in this equation the value of y and the corresponding value of x. Then A_0, A_1, A_2, . . . B_1, B_2, . . . are the unknown quantities to be found, and as many normal equations as there are of these are to be formed from the observational equations in the ordinary manner. This is of importance in the determination of the expressions for tides, electromotive forces of alternators, &c (see p. 659 above).

For further information on this subject the reader may refer to Merriman's treatise on *The Method of Least Squares*, Chauvenet's *Astronomy*, Vol. II., or Airy's *Errors of Observation*.

III. NOTE ON EQUATIONS (80), (81), PAGE 242.

It follows by the equations of electromotive force (33), p. 159, that there must exist in the function from which the electromotive forces are found by Lagrange's equations a term of the form

$$-\tfrac{1}{2}C'\mu\{u(\mathfrak{v}\gamma-\mathfrak{w}\beta)+v(\mathfrak{w}\alpha-\mathfrak{u}\gamma)+w(\mathfrak{u}\beta-\mathfrak{v}\alpha)\}$$

where $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$ are the time integrals, from some zero of reckoning of the components u, v, w, of the total current (displacement current + conduction current).

If the medium is an insulator $\mathfrak{u}, \mathfrak{v}, \mathfrak{w} = f, g, h$, and the term is

$$-\tfrac{1}{2}C'\mu\{\dot{f}(g\gamma-h\beta)+\dot{g}(h\alpha-f\gamma)+\dot{h}(f\beta-g\alpha)\}.$$

Denoting the term by L we find for the x-component of electromotive force depending upon it in the first case

$$-\frac{d}{dt}\frac{\partial L}{\partial u}+\frac{\partial L}{\partial \mathfrak{u}}=C'\mu(v\gamma-w\beta)+\tfrac{1}{2}C'\mu(\mathfrak{v}\dot{\gamma}-\mathfrak{w}\dot{\beta}) \quad . \quad (a)$$

and in the second

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{f}}+\frac{\partial L}{\partial f}=C'\mu(\dot{g}\gamma-\dot{h}\beta)+\tfrac{1}{2}C'\mu(g\dot{\gamma}-h\dot{\beta}) \quad . \quad (b)$$

and similarly for the other components.*

The first term in each case refers to the Hall effect, the second to an electromotive force which has not been observed. An estimate of the amount of this force is given by J. J. Thomson (*Applications of Dynamics*, &c.) which shows that it is probably much too small to be ever observed.

In the case of an insulator the resultant electric force due to the second term in (b) is at right angles to the plane through the direction of the electric force and that in which the magnetic force varies most rapidly with the time. The magnitude of the force is $\tfrac{1}{2}C'\mu\dot{\mathbf{H}}\mathbf{D}\sin\theta$, where $\dot{\mathbf{H}}$ is the greatest time-rate of change of the magnetic force, $\mathbf{D}$ the electric displacement, and θ the angle between $\dot{\mathbf{H}}$ and $\mathbf{D}$.

* The existence of this term was first pointed out by Prof. Fitzgerald, *Phil. Trans.* 1880, Part II.

The electromotive forces in (*a*) and (*b*) fall to be added to the components given at p. 191, to render these more complete. It is not impossible that there may be other terms in the Lagrangian function as yet unrevealed.

IV. NOTE ON ψ, PAGE 192.

The statement in lines 4, 5, 6, that "ψ may be taken as the potential of any electrostatic distribution which may exist in the field," requires correction when the equations refer to moving conductors. In this latter case, as has been pointed out by J. J. Thomson (Maxwell's *Electricity and Magnetism*, Vol. II. p. 260), ψ ought to be taken as including both the electrostatic potential and the term $F\dot{x} + G\dot{y} + H\dot{z}$ which is left out of account in the mode of proving the equations of electromotive force adopted at p. 191. This term disappears when integrated round a closed circuit.

V. NOTE ON PAGE 329.

Proof of the equation

$$\phi(x) = \epsilon^{2\sqrt{x}}/(2\sqrt{\pi}x^{\frac{1}{4}})$$

which holds when x is very great.

It is easy to see from p. 330 that

$$\phi(x) = J_0(2i\sqrt{x})$$

if $i = \sqrt{-1}$. Also by the differential equation satisfied by J_0, if $y = \pi J_0(z)$,

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + y = 0.$$

This last equation can be written

$$\frac{d^2(y\sqrt{z})}{dz^2} + \left(\frac{1}{4z^2} + 1\right)y\sqrt{z} = 0,$$

which when z is very great becomes

$$\frac{d^2(y\sqrt{z})}{dz^2} + y\sqrt{z} = 0.$$

A solution of this is of course

$$y\sqrt{z} = A\cos z + B\sin z,$$

and (Todhunter, *The Functions of Laplace*, &c., p. 311, it can be shown that $A = B = \sqrt{\pi}$.

If now we put $2i\sqrt{x}$ for z in the last result, we find since $\sqrt{i} = (1+i)/\sqrt{2}$,

$$\sqrt{\pi}(1+i)x^{\frac{1}{4}}\phi(x) = \cos 2i\sqrt{x} + \sin 2i\sqrt{2}.$$

When exponential values are substituted for $\cos 2i\sqrt{x}$ and $\sin 2i\sqrt{x}$, this gives

$$\sqrt{\pi}x^{\frac{1}{4}}\phi(x) = \tfrac{1}{2}$$

or

$$\phi(x) = \epsilon^{2\sqrt{x}}(2\sqrt{\pi}x^{\frac{1}{4}}).$$

VI. Table for the Calculation of the Mutual Inductance M of Two Coaxial Circles of Radii a, a', and Distance apart b.

Calculated for intervals of 6′ in the value of

$$\cos^{-1}\{\sqrt{(a-a')^2+b^2}/\sqrt{(a+a')^2+b^2}\}$$ from 60° to 90°.

	$\log_{10}\dfrac{M}{4\pi\sqrt{aa'}}$		$\log_{10}\dfrac{M}{4\pi\sqrt{aa'}}$		$\log_{10}\dfrac{M}{4\pi\sqrt{aa'}}$
60° 0′	$\bar{1}$·4994783	64° 0′	$\bar{1}$·6101472	68° 0′	$\bar{1}$·7203003
6′	$\bar{1}$·5022651	6′	$\bar{1}$·6128998	6′	$\bar{1}$·7230640
12′	$\bar{1}$·5050505	12′	$\bar{1}$·6156522	12′	$\bar{1}$·7258286
18′	$\bar{1}$·5078345	18′	$\bar{1}$·6184042	18′	$\bar{1}$·7285942
24′	$\bar{1}$·5106173	24′	$\bar{1}$·6211560	24′	$\bar{1}$·7313609
30′	$\bar{1}$·5133989	30′	$\bar{1}$·6239076	30′	$\bar{1}$·7341287
36′	$\bar{1}$·5161791	36′	$\bar{1}$·6266589	36′	$\bar{1}$·7368975
42′	$\bar{1}$·5189582	42′	$\bar{1}$·6294101	42′	$\bar{1}$·7396675
48′	$\bar{1}$·5217361	48′	$\bar{1}$·6321612	48′	$\bar{1}$·7424387
54′	$\bar{1}$·5245128	54′	$\bar{1}$·6349121	54′	$\bar{1}$·7452111
61° 0′	$\bar{1}$·5272883	65° 0′	$\bar{1}$·6376629	69° 0′	$\bar{1}$·7479848
6′	$\bar{1}$·5300628	6′	$\bar{1}$·6404137	6′	$\bar{1}$·7507597
12′	$\bar{1}$·5328361	12′	$\bar{1}$·6431645	12′	$\bar{1}$·7535361
18′	$\bar{1}$·5356084	18′	$\bar{1}$·6459153	18′	$\bar{1}$·7563138
24′	$\bar{1}$·5383796	24′	$\bar{1}$·6486660	24′	$\bar{1}$·7590929
30′	$\bar{1}$·5411498	30′	$\bar{1}$·6514169	30′	$\bar{1}$·7618735
36′	$\bar{1}$·5439190	36′	$\bar{1}$·6541678	36′	$\bar{1}$·7646556
42′	$\bar{1}$·5466872	42′	$\bar{1}$·6569189	42′	$\bar{1}$·7674392
48′	$\bar{1}$·5494545	48′	$\bar{1}$·6596701	48′	$\bar{1}$·7702245
54′	$\bar{1}$·5522209	54′	$\bar{1}$·6624215	54′	$\bar{1}$·7730114
62° 0′	$\bar{1}$·5549864	66° 0′	$\bar{1}$·6651732	70° 0′	$\bar{1}$·7758000
6′	$\bar{1}$·5577510	6′	$\bar{1}$·6679250	6′	$\bar{1}$·7785903
12′	$\bar{1}$·5605147	12′	$\bar{1}$·6706772	12′	$\bar{1}$·7813823
18′	$\bar{1}$·5632776	18′	$\bar{1}$·6734296	18′	$\bar{1}$·7841762
24′	$\bar{1}$·5660398	24′	$\bar{1}$·6761824	24′	$\bar{1}$·7869720
30′	$\bar{1}$·5688011	30′	$\bar{1}$·6789356	30′	$\bar{1}$·7897696
36′	$\bar{1}$·5715618	36′	$\bar{1}$·6816891	36′	$\bar{1}$·7925692
42′	$\bar{1}$·5743217	42′	$\bar{1}$·6844431	42′	$\bar{1}$·7953709
48′	$\bar{1}$·5770809	48′	$\bar{1}$·6871976	48′	$\bar{1}$·7981745
54′	$\bar{1}$·5798394	54′	$\bar{1}$·6899526	54′	$\bar{1}$·8009803
63° 0′	$\bar{1}$·5825971	67° 0′	$\bar{1}$·6927081	71° 0′	$\bar{1}$·8037882
6′	$\bar{1}$·5853546	6′	$\bar{1}$·6954642	6′	$\bar{1}$·8065983
12′	$\bar{1}$·5881113	12′	$\bar{1}$·6982209	12′	$\bar{1}$·8094107
18′	$\bar{1}$·5908675	18′	$\bar{1}$·7009782	18′	$\bar{1}$·8122253
24′	$\bar{1}$·5936231	24′	$\bar{1}$·7037362	24′	$\bar{1}$·8150423
30′	$\bar{1}$·5963782	30′	$\bar{1}$·7064949	30′	$\bar{1}$·8178617
36′	$\bar{1}$·5991322	36′	$\bar{1}$·7092544	36′	$\bar{1}$·8206836
42′	$\bar{1}$·6018871	42′	$\bar{1}$·7120146	42′	$\bar{1}$·8235080
48′	$\bar{1}$·6046408	48′	$\bar{1}$·7147756	48′	$\bar{1}$·8263349
54′	$\bar{1}$·6073942	54′	$\bar{1}$·7175375	54′	$\bar{1}$·8291645

TABLE FOR THE CALCULATION OF THE MUTUAL INDUCTANCE M OF TWO COAXIAL CIRCLES OF RADII a, a'—*continued.*

	$\log_{10} \frac{M}{4\pi\sqrt{aa'}}$		$\log_{10} \frac{M}{4\pi\sqrt{aa'}}$		$\log_{10} \frac{M}{4\pi\sqrt{aa'}}$
72° 0′	$\bar{1}$·8319967	76° 0′	$\bar{1}$·9482196	80° 0′	·0741816
6′	$\bar{1}$·8348316	6′	$\bar{1}$·9512205	6′	·0775316
12′	$\bar{1}$·8376693	12′	$\bar{1}$·9542272	12′	·0808944
18′	$\bar{1}$·8405099	18′	$\bar{1}$·9572400	18′	·0842702
24′	$\bar{1}$·8433534	24′	$\bar{1}$·9602590	24′	·0876592
30′	$\bar{1}$·8461998	30′	$\bar{1}$·9632841	30′	·0910619
36′	$\bar{1}$·8490493	36′	$\bar{1}$·9663157	36′	·0944784
42′	$\bar{1}$·8519018	42′	$\bar{1}$·9693537	42′	·0979091
48′	$\bar{1}$·8547575	48′	$\bar{1}$·9723983	48′	·1013542
54′	$\bar{1}$·8576164	54′	$\bar{1}$·9754497	54′	·1048142
73° 0′	$\bar{1}$·8604785	77° 0′	$\bar{1}$·9785079	81° 0′	·1082893
6′	$\bar{1}$·8633440	6′	$\bar{1}$·9815731	6′	·1117799
12′	$\bar{1}$·8662129	12′	$\bar{1}$·9846454	12′	·1152863
18′	$\bar{1}$·8690852	18′	$\bar{1}$·0877249	18′	·1188089
24′	$\bar{1}$·8719611	24′	$\bar{1}$·9908118	24′	·1223481
30′	$\bar{1}$·8748406	30′	$\bar{1}$·9939062	30′	·1259043
36′	$\bar{1}$·8777237	36′	$\bar{1}$·9970082	36′	·1294778
42′	$\bar{1}$·8806106	42′	·0001181	42′	·1330691
48′	$\bar{1}$·8835013	48′	·0032359	48′	·1366786
54′	$\bar{1}$·8863958	54′	·0063618	54′	·1403067
74° 0′	$\bar{1}$·8892943	78° 0′	·0094959	82° 0′	·1439539
6′	$\bar{1}$·8921969	6′	·0126385	6′	·1476207
12′	$\bar{1}$·8951036	12′	·0157896	12′	·1513075
18′	$\bar{1}$·8980144	18′	·0189494	18′	·1550149
24′	$\bar{1}$·9009295	24′	·0221181	24′	·1587434
30′	$\bar{1}$·9038489	30′	·0252959	30′	·1624935
36′	$\bar{1}$·9067728	36′	·0284830	36′	·1662658
42′	$\bar{1}$·9097012	42′	·0316794	42′	·1700609
48′	$\bar{1}$·9126341	48′	·0348855	48′	·1738794
54′	$\bar{1}$·9155717	54′	·0381014	54′	·1777219
75° 0′	$\bar{1}$·9185141	79° 0′	·0413273	83° 0′	·1815890
6′	$\bar{1}$·9214613	6′	·0445633	6′	·1854815
12′	$\bar{1}$·9244135	12′	·0478098	12′	·1894001
18′	$\bar{1}$·9273707	18′	·0510668	18′	·1933455
24′	$\bar{1}$·9303330	24′	·0543347	24′	·1973184
30′	$\bar{1}$·9333005	30′	·0576136	30′	·2013197
36′	$\bar{1}$·9362733	36′	·0609037	36′	·2053502
42′	$\bar{1}$·9392515	42′	·0642054	42′	·2094108
48′	$\bar{1}$·9422352	48′	·0675187	48′	·2135026
54′	$\bar{1}$·9452246	54′	·0708441	54′	·2176259

TABLE FOR THE CALCULATION OF THE MUTUAL INDUCTANCE M OF TWO COAXIAL CIRCLES OF RADII a, a'—*continued.*

	$\log_{10} \frac{M}{4\pi\sqrt{aa'}}$		$\log_{10} \frac{M}{4\pi\sqrt{aa'}}$		$\log_{10} \frac{M}{4\pi\sqrt{aa'}}$
84° 0′	·2217823	86° 0′	·3139097	88° 0′	·4385420
6′	·2259728	6′	·3191092	6′	·4465341
12′	·2301983	12′	·3243843	12′	·4548064
18′	·2344600	18′	·3297387	18′	·4633880
24′	·2387591	24′	·3351762	24′	·4723127
30′	·2430970	30′	·3407012	30′	·4816206
36′	·2474748	36′	·3463184	36′	·4913595
42′	·2518940	42′	·3520327	42′	·5015870
48′	·2563561	48′	·3578495	48′	·5123738
54′	·2608626	54′	·3637749	54′	·5238079
85° 0′	·2654152	87° 0′	·3698153	89° 0′	·5360007
6′	·2700156	6′	·3759777	6′	·5490969
12′	·2746655	12′	·3822700	12′	·5632886
18′	·2793670	18′	·3887006	18′	·5788406
24′	·2841221	24′	·3952792	24′	·5961320
30′	·2889329	30′	·4020162	30′	·6157370
36′	·2938018	36′	·4089234	36′	·6385907
42′	·2987312	42′	·4160138	42′	·6663883
48′	·3037238	48′	·4233022	48′	·7027765
54′	·3087823	54′	·4308053	54′	·7586941

VII. EFFECTIVE RESISTANCE OF CONDUCTORS CARRYING ALTERNATING CURRENTS.

(For explanation of notation refer to Section III., Chapter VI. above). It can be shown that if C_N be the current at the axis of a wire and C'_N the current at distance r from the axis, for N periods per second, and $\theta = 2\pi Nt$

$$C'_N = C_N(\text{ber}\, q \cos\theta - \text{bei}'\, q \sin\theta).$$

For copper carrying an alternating current of 80 periods per second, the column below headed q may be taken as containing the diameters of the wires; and in respect to the distribution of the current through the wire expressed by the formula above, q may be taken as the diameter of the cylindric shell in which the current density is to be calculated.

TABLE OF NUMERICAL VALUES.

(Calculated for Lord Kelvin by Mr. M. Maclean.)

q	$\dfrac{\text{bei}'\,\text{ber} - \text{ber}'\,\text{bei}}{\text{ber}'^2 + \text{bei}'^2}$	$\dfrac{1}{2}q\,\dfrac{\text{bei}\,\text{ber}' - \text{ber}\,\text{bei}'}{\text{ber}'^2 + \text{bei}'^2}$
0·0	∞	1·0000
0·5	4·0000	1·0000
1·0	2·00014	1·0001
1·5	1·3678	1·0258
2·0	1·0805	1·0805
2·5	·9398	1·1747
3·0	·8787	1·3180
3·5	·8526	1·4920
4·0	·8389	1·6778
4·5	·8270	1·8628
5·0	·8172	2·0430
5·5	·8069	2·2190
6·0	·7979	2·3937
8	·7739	3·0950
10	·7588	3·7940
15	·7431	5·5732
20	·7424	7·3250

From the data here given and some further data supplied by Lord Kelvin, Mr. W. M. Mordey has calculated the following table :—

VIRTUAL RESISTANCE OF CONDUCTORS CARRYING ALTERNATING CURRENTS.

From Mr. W. M. Mordey's paper of Alternate Current Working, Inst. El. Eng., May 23, 1889, *Electrician*, May 31.

Diameter.		Area.		Percentage Increase of Ordinary Resistance.	Number of Complete Periods per Second.
Millimetres.	Inches.	Sq. mm.	Sq. in.		
10	·3937	78·54	·122	Less than $\frac{1}{10}$	80
15	·5905	176·7	·274	2·5	
20	·7874	314·16	·487	8	
25	·9842	490·8	·760	17·5	
40	1·575	1,256	1·95	68	
100	3·937	7,854	12·17	3·8 times	
1000	39 39	785,400	1,217	35 times	
9	·3543	63 62	·098	Less than $\frac{1}{100}$	100
13·4	·5280	141·3	·218	2·5	
18	·7086	254 4	·394	8	
22·4	·8826	394	·611	17·5	
7·75	·3013	47·2	·071	Less than $\frac{1}{100}$	133
11·61	·4570	106	·164	2·5	
15 5	·6102	189	·292	8	
19·36	·7622	294	·456	17 5	

VIII.

RESOLUTIONS OF THE BOARD OF TRADE COMMITTEE ON ELECTRICAL STANDARDS.

(From a Report to the President of the Board of Trade, dated November 29, 1892. These resolutions the committee desire to substitute for those contained in a previous report of date July, 1891, with the view of obtaining international agreement as to Electrical Standards) :—

Resolutions.

(1) "That it is desirable that new denominations of standards for the measurement of electricity should be made and approved by Her Majesty in Council as Board of Trade standards.

(2) "That the magnitudes of these standards should be determined on the electromagnetic system of measurement with reference to the centimetre as unit of length, the gramme as unit of mass, and the second as unit of time, and that by the terms centimetre and gramme are meant the standards of those denominations deposited with the Board of Trade.

(3) "That the standard of electrical resistance should be denominated the ohm, and should have the value 1,000,000,000 in terms of the centimetre and second.

(4) "That the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14·4521 grammes in mass, of a constant cross sectional area, and of a length of 106·3 centimetres, may be adopted as one ohm.

[redacted]

(5) "That a material standard, constructed in solid metal, should be adopted as the standard ohm, and should from time to time be verified by comparison with a column of mercury of known dimensions.

(6) "That for the purpose of replacing the standard, if lost, destroyed, or damaged, or for ordinary use, a limited number of copies should be constructed which should be periodically compared with the standard ohm.

(7) "That resistances constructed in solid metal should be adopted as Board of Trade standards for multiples and submultiples of the ohm.

(8) "That the value of the standard of resistance constructed by a Committee of the British Association for the Advancement of Science in the years 1863 and 1864, and known as the British Association unit, may be taken as ·9866 of the ohm.

(9) "That the standard of electrical current should be denominated the ampere, and should have the value one-tenth (0·1) in terms of the centimetre, gramme, and second.

(10) "That an unvarying current which, when passed through a solution of nitrate of silver in water, in accordance with the specification attached to this report, deposits silver at the rate of 0·001118 of a gramme per second, may be taken as a current of one ampere.*

(11) "That an alternating current of one ampere shall mean a current such that the square root of the time-average of the square of its strength at each instant in amperes is unity.

(12) "That instruments constructed on the principle of the balance, in which, by the proper disposition of the conductors, forces of attraction and repulsion are produced, which depend upon the amount of cur-

* For full particulars as to carrying out the electrolysis of silver, and the construction and use of Clark's cell, see Chapter VII. above.

rent passing, and are balanced by known weights, should be adopted as the Board of Trade standards for the measurement of current whether unvarying or alternating.

(13) "That the standard of electrical pressure should be denominated the volt, being the pressure which, if steadily applied to a conductor whose resistance is one ohm, will produce a current of one ampere.

(14) "That the electrical pressure at a temperature of 15° centigrade between the poles or electrodes of the voltaic cell known as Clark's cell, prepared in accordance with the specification attached to this report, may be taken as not differing from a pressure of 1·434 volts, by more than one part in 1000.*

(15) "That an alternating pressure of one volt shall mean a pressure such that the square root of the time-average of the square of its value at each instant in volts is unity.

(16) "That instruments constructed on the principle of Lord Kelvin's quadrant electrometer used idiostatically, and, for high-pressures, instruments on the principle of the balance, electrostatic forces being balanced against a known weight, should be adopted as Board of Trade standards for the measurement of pressure, whether unvarying or alternating."

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* See note on previous page.

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THE END.

RICHARD CLAY AND SONS, LIMITED, LONDON AND BUNGAY.